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A Batyrev type classification of $\mathbb{Q}$-factorial projective toric varieties

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Abstract: The present paper generalizes, inside the class of projective toric varieties, the classification [2], performed by Batyrev in 1991 for smooth complete toric varieties, to the singular $\mathbb{Q}$-factorial case.

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Introduction

The present paper is the third part of a longstanding tripartite study aimed at realizing, for $\mathbb{Q}$-factorial projective toric varieties, a classification inspired by what Batyrev did in [2] for smooth complete toric varieties. The first part of this study is [22], in which we studied Gale duality from the $\mathbb{Z}$-linear point of view and defined poly weighted spaces (PWS, for short; see Definition 1.4) as $\mathbb{Q}$-factorial complete toric varieties whose classes group is free. The second part is [23], in which we exhibited a canonic covering PWS Y for every $\mathbb{Q}$-factorial complete toric variety X, such that the covering map $Y \to X$ is a torus-equivariant Galois covering, induced by the multiplicative action of the finite group $\mu(X) := \mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*)$ on Y and ramified in codimension at least 2. The reader will often be referred to these papers for notation, preliminaries and results.

Considerably simplifying the situation, we summarize the main results of the present paper as follows:

Theorem 0.1. *Let X be a $\mathbb{Q}$-factorial projective toric variety satisfying* some good conditions *on an associated* weight matrix*; see* [22, Definition 3.9] *and Definition* 1.3 *below. Then X is birational and isomorphic in codimension* 1 *to a finite abelian quotient of a PWS which is a* toric cover (*see Definition* 2.17) *of a* weighted projective toric bundle (WPTB)*; see §* 2.2.1.

Moreover X is isomorphic to a finite abelian quotient of a PWS which is a toric cover of a WPTB if and only if its fan is associated with a chamber of the secondary fan which is maximally bordering (*maxbord, see Definition* 2.5) *inside the Gale dual* (*or GKZ*) *cone* $\mathcal{Q}$.

Finally X is isomorphic to a finite abelian quotient of a PWS produced from a toric cover of a weighted projective space (*WPS*) *by a sequence of toric covers of weighted projective toric bundles if and only if its fan chamber is* recursively maxbord (*see Definition* 2.27) *inside the Gale dual cone* $\mathcal{Q}$.

In any case, the finite abelian quotient is trivial if and only if $\mathrm{Cl}(X)$ *is a free abelian group meaning that X is a PWS; recall* [23, Theorem 2.1].

This statement is a patching of Theorems 3.4, 3.7 and 3.8, which by [23, Theorem 2.2] are immediate consequences of Theorems 2.22, 2.24 and 2.28, respectively.

Before clarifying the meaning of emphasized terms in the statement above, we mention that results of this kind are well known in the context of smooth complete toric varieties. From this point of view, the first important result is probably the Kleinschmidt classification [16] of smooth projective toric varieties with Picard

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number (in the following called *rank*) $r \leq 2$ as suitable projective toric bundles (PTB, for short) over a projective space of smaller dimension. Later Kleinschmidt and Sturmfels [17] proved that every smooth complete toric variety of rank $r \leq 3$ is necessarily projective, thus extending the Kleinschmidt classification to the range of smooth complete toric variety of rank $r \leq 2$. In 1991 Batyrev generalized the Kleinschmidt classification by introducing the concepts of *primitive collection* and of associated *primitive relation* (see [2, Definitions 2.6,7,8] and the following § 2.1): he proved that a smooth complete toric variety $X(\Sigma)$ is a PTB over a toric variety of smaller dimension if and only if the fan Σ admits a primitive collection with focus 0 (in the following also called *nef*: see 2.1) which is disjoint from any other primitive collection of Σ; see [2, Proposition 4.1]. Consequently a smooth complete toric variety $X(\Sigma)$ is produced from a projective space by a sequence of PTB if and only if Σ is a *splitting fan*; see [2, Definition 4.2, Theorem 4.3, Corollary 4.4].

We emphasize that Batyrev's techniques are deeply connected with the smoothness hypothesis. In fact, the starting step of the induction proving [2, Theorem 4.3] does not hold in the singular set up, even for projective varieties: there exist projective $\mathbb{Q}$-factorial toric varieties, of rank $r \geq 2$, not admitting any numerically effective primitive collection, although all their primitive collections are disjoint pair by pair. Example 2.31 gives an account of this situation. Even for rank $r = 1$ the singular case appears to be significantly more intricate than the smooth one, since the former necessarily involves some finite covering: on the one hand the unique smooth complete toric variety with $r = 1$ is given by the projective space, on the other hand a $\mathbb{Q}$-factorial complete toric variety with $r = 1$ is a quotient of a weighted projective space (WPS, for short), as proved by Batyrev and Cox [3] and by Conrads [7].

As mentioned, the latter result has been extended to every rank r by [23, Theorem 2.2], here recalled by Theorem 3.2, allowing us to reduce the classification of $\mathbb{Q}$-factorial complete toric varieties to classifying their covering PWS, i.e. to classifying $\mathbb{Q}$-factorial complete toric varieties with free classes group.

Bypassing counterexample 2.31 means characterizing those PWS admitting a nef primitive collection. This is done by stressing remarks of Casagrande [5] and of Cox and von Renesse [10], revising the original Batyrev definition of primitive relation: § 2.1 is largely devoted to this purpose. The idea is that of dually thinking of the numerical class of a primitive relation as a hyperplane in $\mathrm{Cl}(X) \otimes \mathbb{R}$, which we call the *supporting hyperplane* of the primitive collection (see Definition 2.1). By applying the $\mathbb{Z}$-linear Gale duality developed in [22], in § 1.2 a linear algebraic interpretation of the secondary (or GKZ) fan is proposed. More precisely, given an F-matrix V (see Definition 1.2) we can choose a Gale dual W-matrix $Q = \mathcal{G}(V)$ (see Definition 1.3 and [22, § 3.1]) such that Q is a positive and in row echelon form (REF) matrix (see [22, Theorem 3.18] and the following Proposition 1.6). The secondary fan can then be thought of as a suitable fan whose support is given by the strongly convex cone $\mathcal{Q} = \langle Q \rangle$, called the *Gale dual* cone and generated by the columns of the weight matrix Q. This gives a $\mathbb{Z}$-linear algebraic interpretation of the duality linking simplicial fans generated by the columns of the fan matrix V and *bunches of cones*, in the sense of [4], inside the Gale dual cone $\mathcal{Q}$, in terms of the $\mathbb{Z}$-linear Gale duality linking submatrices of V and Q exhibited by [22, Theorem 3.2]: in particular, this gives a bijection between simplicial fans Σ giving $\mathbb{Q}$-factorial projective toric varieties X whose fan matrix is V and r-dimensional subcones γ of $\mathcal{Q}$ (called *chambers*) obtained as intersection of the cones in the corresponding bunch of cones. In particular, $\mathcal{Q}$ turns out to be contained in the positive orthant F^r_+ of $\mathrm{Cl}(X) \otimes \mathbb{R}$ and the properties of a primitive collection $\mathcal{P}$ for Σ can be thought of in terms of mutual position of the corresponding support hyperplane $H_{\mathcal{P}}$ with respect to the fan chamber γ (see Propositions 2.2 and 2.3). E.g. $\mathcal{P}$ is numerically effective if and only if $H_{\mathcal{P}}$ cuts out a facet of the Gale dual cone $\mathcal{Q}$, i.e. $\mathcal{P}$ is a *bordering* primitive collection in the sense of Definition 2.5. Moreover a chamber $\gamma \subseteq \mathcal{Q}$ is called (*maximally*) *bordering* if it admits a (facet) face lying on the boundary $\partial\mathcal{Q}$. Theorem 2.11 exhibits the relation between bordering chambers and bordering primitive collections, thus characterizing those PWS admitting a nef primitive collection we are looking for: it is the generalization of [2, Proposition 3.2] to the singular $\mathbb{Q}$-factorial set up. Then extension of the Batyrev classification to the singular $\mathbb{Q}$-factorial case is given by Theorem 2.22: in particular the latter together with Proposition 2.25 generalizes [2, Proposition 4.1], together with Proposition 2.26 generalizes [2, Theorem 4.3] and together with Theorem 2.28 generalizes [2, Corollary 4.4].

Now we explain a hypothesis in the statement of Theorem 0.1 above: *good conditions on the associated weight matrix* means that Q can be set in a positive REF such that, by deleting the bottom row and the last $s+1$ columns on the right, we still get an (almost) W-matrix Q' which gives a weight matrix of the s-fibration basis.

In other words, this condition means that the Gale dual cone $\mathcal{Q}$ may admit a maximally bordering fan chamber, which is the generalization of Batyrev's condition requiring the existence of a nef primitive collection disjoint from any further primitive collection of the same fan.

From the geometric point of view, a maximally bordering chamber corresponds to giving a fibering morphism whose fibers are a suitable abelian quotient of a weighted projective space (called a *fake WPS*): this is a well known fact which is essentially rooted in Reid's work [20]. See also [15, Proposition 1.11] and [6, § 2] for more recent results suggesting possible interesting applications, of techniques here presented, in the more general setup of Mori Dream Spaces. As explained in § 2.4, Remark 2.39 and Remark 3.5, this fibering morphism gives the Stein factorization of the toric cover of a WPTB exhibited by Theorem 2.22, and more generally by the previous Theorem 0.1, thus obtaining a commutative diagram

$$\begin{tikzcd} X(\Sigma) \arrow[r, "\text{finite}", "f"'] \arrow[d, "\phi", "\substack{\text{fake WPS}\\ \text{fibering}}"'] & \mathbb{P}^W(\mathcal{E}) \arrow[d, "\varphi"', "\text{WPTB}"] \\ X_0(\Sigma_0) \arrow[r, "f_0", "\text{finite}"'] & X'(\Sigma') \end{tikzcd}$$

whose vertical morphisms have connected fibers and whose horizontal ones are finite morphisms of toric varieties. Note that if X is smooth then both the finite toric morphisms f and f_0 are trivial giving that $\phi = \varphi$ is precisely Batyrev's projective toric bundle. The right hand side factorization $\varphi \circ f$ has the great advantage of being constructively described, giving a procedural approach to an effective determination of all the morphisms and varieties involved, as examples in § 2.7 show. The last procedure can be easily implemented in any computer algebra package (we used Maple to perform all the necessary computations).

Now we describe the structure of this paper and summarize the further obtained results. § 1 introduces notation and preliminaries: the list in § 1.1 recalls symbols defined in [22] and [23], and § 1.2 introduces the above mentioned $\mathbb{Z}$-linear algebraic interpretation of the secondary fan. Theorem 1.8 bridges between the linear algebraic secondary fan defined in Definition 1.7 and the usual secondary fan of a $\mathbb{Q}$-factorial complete toric variety. The bijection and $\mathbb{Z}$-linear Gale duality between projective fans and GKZ chambers are established by Theorem 1.9.

The long § 2 is the main part of the present paper, in which the Batyrev-type classification of PWS is performed. § 2.1 revises the concept of a primitive collection and introduces the bordering notion for collections and chambers with respect to the Gale dual cone $\mathcal{Q}$. In § 2.2 we introduce the main ingredients for the classification. § 2.2.1 defines a *weighted projective toric bundle* (WPTB) $\mathbb{P}^W(\mathcal{E})$ as the Proj of the W-weighted symmetric algebra $S^W(\mathcal{E})$ over a locally free sheaf $\mathcal{E}$. In Proposition 2.16 we describe the fan of a WPTB, as a $\mathbb{Q}$-factorial toric variety, along the lines of what is done in [9, Proposition 7.3.3] for a projective toric bundle (PTB). § 2.2.2 recalls the concept of a *toric cover*, as defined in [1]. § 2.3 is the core of the present paper with Theorem 2.22 and Theorem 2.24, from the birational point of view (i.e. up to toric flips as defined in § 1.3). The geometric meaning of a maxbord chamber is explained in § 2.4. In § 2.5 generalize, in the singular $\mathbb{Q}$-factorial setting, Batyrev's concept of a splitting fan, giving rise to Theorem 2.28. In particular, when $r \leq 3$, Theorem 2.33 and Remark 2.30 give a partial extension to the singular $\mathbb{Q}$-factorial case of Batyrev's results on the number of primitive relations; see [2, § 5 and 6]. § 2.6 gives a partial generalization, to the $\mathbb{Q}$-factorial set up, of results about contractible classes on smooth projective toric varieties due to Casagrande [5] and Sato [27]: our study is limited to the case of numerically effective classes (see Proposition 2.36 and Theorem 2.38). Subsection § 2.7 treats applications of all the techniques described, by means of five examples: here it is rather important for the reader to be equipped with some computer algebra package which has the ability to produce Hermite and Smith normal forms of matrices and their switching matrices. For example, using Maple, similar procedures are given by `HermiteForm` and `SmithForm` with their output options.

Note that the last Example 2.44 exhibits the case of a (4-dimensional) $\mathbb{Q}$-factorial complete toric variety of Picard number $r = 3$ whose Nef cone is 0, i.e. which does not admit any non-trivial numerical effective divisor: we think this is a significant and new example since in the smooth case Fujino and Payne [11] proved that this is not possible for $r \leq 4$, at least for dimension ≤ 3. For further considerations about this subject see Remark 2.45.

§ 3 applies results obtained in § 2 for PWS to the case of a general $\mathbb{Q}$-factorial projective toric variety. The above Theorem 0.1 is the patching of results stated there. This section ends with a further example aimed at classifying a $\mathbb{Q}$-factorial projective variety which is not a PWS.

1 Preliminaries and notation

On one hand, the present paper is a further application of the $\mathbb{Z}$-linear Gale Duality developed in [22], to which the reader is referred for notation and preliminary results. In particular, for notation on toric varieties, cones and fans, the reader is referred to [22, § 1.1], and for linear algebraic preliminaries about normal forms of matrices (Hermite and Smith normal forms, HNF and SNF for short) to [22, § 1.2]. $\mathbb{Z}$-linear Gale Duality, fan matrices (F-matrices) and weight matrices (W-matrices) are developed in [22, § 3]. On the other hand, the results presented here are consequences of the fact that a $\mathbb{Q}$-factorial complete toric variety X is always a finite geometric quotient of a poly weighted space (PWS) Y, which turns out to be the *universal* 1*-connected in codimension* 1 *covering* (1*-covering*) of X; see [23, Definition 1.5, Theorem 2.2].

Every time the needed nomenclature will be recalled either directly by giving the definition or by giving a reference. Here is a list of main notation and relevant references:

1.1 List of notation. Let $X(\Sigma)$ be an n-dimensional toric variety and let $T \cong (\mathbb{C}^*)^n$ be the acting torus.

- $M, N, M_\mathbb{R}, N_\mathbb{R}$ denote the *group of characters* of T, its dual group and their tensor products with $\mathbb{R}$, respectively;
- $\Sigma \subseteq N_\mathbb{R}$ is the fan defining X;
- $\Sigma(i)$ is the *i-skeleton* of Σ, which is the collection of all the i-dimensional cones in Σ;
- $|\Sigma|$ is the *support* of the fan Σ, i.e. $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_\mathbb{R}$;
- $\det(\sigma) := |\det(V_\sigma)|$ for a simplicial cone $\sigma \in \Sigma(n)$ whose primitive generators give the columns of V_σ;
- σ is *unimodular* if $\det(\sigma) = 1$;
- $r = \mathrm{rk}(X)$ is the Picard number of X, also called the *rank* of X;
- $\mathfrak{P} = \mathfrak{P}(1, \dots, n + r)$ is the power set of the set $\{1, \dots, n + r\}$ of indices;
- $F^r_\mathbb{R} \cong \mathbb{R}^r$ is the $\mathbb{R}$-linear span of the free part of $\mathrm{Cl}(X(\Sigma))$;
- F^r_+ is the positive orthant of $F^r_\mathbb{R} \cong \mathbb{R}^r$;
- $\langle \mathbf{v}_1, \dots, \mathbf{v}_s\rangle \subseteq \mathbb{N}_\mathbb{R}$ denotes the cone generated by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_s \in N_\mathbb{R}$;
- if $s = 1$ then this cone is also called the *ray* generated by $\mathbf{v}_1$;
- $\mathcal{L}(\mathbf{v}_1, \dots, \mathbf{v}_s) \subseteq N$ denotes the sublattice spanned by $\mathbf{v}_1, \dots, \mathbf{v}_s \in N$.

Let $A \in \mathbf{M}(d, m; \mathbb{Z})$ be a $d \times m$ integer matrix; then

- $\mathcal{L}_r(A) \subseteq \mathbb{Z}^m$ denotes the sublattice spanned by the rows of A;
- $\mathcal{L}_c(A) \subseteq \mathbb{Z}^d$ denotes the sublattice spanned by the columns of A;
- A_I, A^I for $I \subseteq \{1, \dots, m\}$: the former is the submatrix of A given by the columns indexed by I and the latter is the submatrix of A whose columns are indexed by the complementary subset $\{1, \dots, m\}\backslash I$;
- ${}_sA$, ${}^sA$ for $1 \le s \le d$; the former is the submatrix of A given by the lower s rows and the latter is the submatrix of A given by the upper s rows of A;
- $\mathrm{HNF}(A)$ and $\mathrm{SNF}(A)$ denote the Hermite and the Smith normal forms of A, respectively;
- REF Row Echelon Form of a matrix;
- *positive* (≥ 0): a matrix (vector) whose entries are non-negative.
- *strictly positive* (> 0): a matrix (vector) whose entries are strictly positive.

Given an F-matrix $V = (\mathbf{v}_1, \dots, \mathbf{v}_{n+r}) \in \mathbf{M}(n, n + r; \mathbb{Z})$, see Definition 1.2 below, then

- $\langle V\rangle = \langle \mathbf{v}_1, \dots, \mathbf{v}_{n+r}\rangle \subseteq N_{\mathbb{R}}$ denotes the cone generated by the columns of V;
- $\mathcal{SF}(V) = \mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_{n+r})$ is the set of all rational simplicial fans Σ such that $\Sigma(1) = \{\langle \mathbf{v}_1\rangle, \dots, \langle \mathbf{v}_{n+r}\rangle\} \subset N_{\mathbb{R}}$, see [22, Definition 1.3];
- $\mathbb{P}\mathcal{SF}(V) := \{\Sigma \in \mathcal{SF}(V) \mid X(\Sigma) \text{ is projective}\}$;
- $\mathcal{G}(V) = Q$ is a *Gale dual* matrix of V, see [22, § 3.1];
- $\mathcal{Q} = \langle \mathcal{G}(V)\rangle \subseteq F^r_+$ is a *Gale dual cone* of $\langle V\rangle$: it is always assumed to be generated in $F^r_{\mathbb{R}}$ by the columns of a positive REF matrix $Q = \mathcal{G}(V)$, see Proposition 1.6 below.
- V^{red} is the *reduced* matrix of V, see [22, Definition 3.13], whose columns are given by the primitive generators of $\langle \mathbf{v}_i\rangle$, with $1 \leq i \leq n+r$.
- $Q^{\mathrm{red}} = \mathcal{G}(V^{\mathrm{red}})$ is the *reduced* matrix of $Q = \mathcal{G}(V)$, see [22, Definition 3.14].

We recall four fundamental definitions:

Definition 1.1. An n-dimensional $\mathbb{Q}$-factorial complete toric variety $X = X(\Sigma)$ of rank r is the toric variety defined by an n-dimensional *simplicial* and *complete* fan Σ such that $|\Sigma(1)| = n+r$; see [22, § 1.1.2]. In particular the rank r coincides with the Picard number, i.e. $r = \mathrm{rk}(\mathrm{Pic}(X))$.

Definition 1.2 ([22], Definition 3.10). An *F-matrix* is an $n \times (n+r)$ matrix V with integer entries such that

a) $\mathrm{rk}(V) = n$;
b) V is *F-complete*, i.e. $\langle V\rangle = N_{\mathbb{R}} \cong \mathbb{R}^n$, see [22, Definition 3.4];
c) all the columns of V are non-zero;
d) if $\mathbf{v}$ is a column of V, then V does not contain another column of the form $\lambda\mathbf{v}$ where $\lambda > 0$ is real number.

A *CF-matrix* is an F-matrix satisfying the further requirement

e) the sublattice $\mathcal{L}_c(V) \subset \mathbb{Z}^n$ is cotorsion free, which means that $\mathcal{L}_c(V) = \mathbb{Z}^n$ or, equivalently, $\mathcal{L}_r(V) \subset \mathbb{Z}^{n+r}$ is cotorsion free.

An F-matrix V is called *reduced* if every column of V is composed by coprime entries, see [22, Definition 3.13].

The most significant example of an F-matrix is given by a matrix V whose columns are integral vectors generating the rays of the 1-skeleton $\Sigma(1)$ of a rational fan Σ. In the following a similar matrix V is called a *fan matrix* of Σ; when every column of V is composed by coprime entries, it is called a *reduced fan matrix*.

Definition 1.3. [22, Definition 3.9] A *W-matrix* is an $r \times (n+r)$ matrix Q with integer entries such that

a) $\mathrm{rk}(Q) = r$;
b) $\mathcal{L}_r(Q)$ has not cotorsion in $\mathbb{Z}^{n+r}$;
c) Q is *W-positive*, which means that $\mathcal{L}_r(Q)$ admits a basis consisting of positive vectors (see list 1.1 and [22, Definition 3.4]);
d) every column of Q is non-zero;
e) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, \dots, 0, 1, 0, \dots, 0)$;
f) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, a, 0, \dots, 0, b, 0, \dots, 0)$ with $ab < 0$.

A W-matrix is called *reduced* if $V = \mathcal{G}(Q)$ is a reduced F-matrix; see [22, Definition 3.13, Theorem 3.15].

In the following, if V is a fan matrix of a rational fan Σ, then $Q = \mathcal{G}(V)$ is called a *weight matrix* of Σ. If V is reduced, then Q is a called a *reduced weight matrix*.

Definition 1.4 ([22] § 2.2). A *poly weighted space* (PWS) is an n-dimensional $\mathbb{Q}$-factorial complete toric variety $Y(\widehat{\Sigma})$ of rank r whose *reduced fan matrix* $\widehat{V}$ (see [22, Definition 3.13]) is a CF-matrix, i.e. if

- $\widehat{V}$ is an $n \times (n+r)$ CF-matrix, and
- $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$.

Recall that a $\mathbb{Q}$-factorial complete toric variety Y is a PWS if and only if it is 1-*connected in codimension* 1 (or simply 1-*connected*): since Y is normal it is equivalent to asking that $\pi_1(Y_{\text{reg}}) \cong \text{Tors}(\text{Cl}(Y)) = 0$, see [23, Corollary 1.8, Theorem 2.1], where $Y_{\text{reg}} \subseteq Y$ is the Zariski open subset of regular points.

Example 1.5. In order to explain the introduced notation, consider a smooth and complete toric variety $X(\Sigma)$, of dimension and rank equal to 3, with reduced fan matrix V given by

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

i.e. such that $\Sigma \in \mathcal{SF}(V)$. One can check that V supports only two complete and simplicial rational fans admitting every column of V as a ray generator; that is $\mathcal{SF}(V) = \{\Sigma_1, \Sigma_2\}$, where Σ_1 and Σ_2 are the fans of cones obtained as all possible faces of the following lists of maximal cones:

$$\Sigma_1(3) = \{\{1,4,5\},\{1,3,5\},\{2,4,5\},\{2,3,5\},\{2,4,6\},\{2,3,6\},\{1,4,6\},\{1,3,6\}\}$$
$$\Sigma_2(3) = \{\{1,4,5\},\{1,3,5\},\{2,4,5\},\{2,3,5\},\{1,2,4\},\{2,3,6\},\{1,3,6\},\{1,2,6\}\}$$

(here a maximal simplicial cone $\langle V_I\rangle$ is identified with the subset of column indexes $I \subseteq \{1, \dots, n+r\}$). Both Σ_1 and Σ_2 are smooth, giving two possible choices for $X(\Sigma)$. Moreover [17] guarantees that those fans are both projective, that is $\mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V)$. A weight matrix of X is given by the choice of a Gale dual matrix of V

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \mathcal{G}(V).$$

Both V and Q are reduced and V is even a CF-matrix, hence X is a PWS.

1.2 The secondary fan. We introduce here a linear algebraic interpretation of the *secondary* (or *GKZ*) fan of a toric variety X. For further details about the secondary fan of a toric variety $X(\Sigma)$, we refer to the comprehensive monograph [9] and its references: among them let us recall the original sources [14], [13] and [19].

Let $V = (\mathbf{v}_1, \dots, \mathbf{v}_{n+r})$ be a reduced F-matrix and $Q := \mathcal{G}(V) = (\mathbf{q}_1, \dots, \mathbf{q}_{n+r})$ an associated W-matrix. Consider the cone generated by the columns of Q

$$\mathcal{Q} = \langle Q\rangle := \langle \mathbf{q}_1, \dots, \mathbf{q}_{n+r}\rangle.$$

For every $\Sigma \in \mathcal{SF}(V)$, one gets $|\Sigma| = \langle V\rangle = N_{\mathbb{R}}$. Then $\mathcal{Q}$ turns out to be a strongly convex cone in $F^r_{\mathbb{R}} := F^r \otimes \mathbb{R}$, where $F^r = \text{Free}(\text{Cl}(X(\Sigma))) \cong \mathbb{Z}^r$, see [9, Lemma 14.3.2]. Recalling [22, Theorems 3.8, 3.18] we can improve this:

Proposition 1.6. *Let F^r_+ denote the positive orthant of $F^r_{\mathbb{R}}$. Then $\langle V\rangle = N_{\mathbb{R}}$ if and only if there exists a positive* REF-*matrix Q such that $Q = \mathcal{G}(V)$ and $\mathcal{Q} = \langle Q\rangle \subset F^r_+$. In particular, for every $\Sigma \in \mathcal{SF}(V)$, $X = X(\Sigma)$ is complete if and only if there exists a positive* REF-*matrix Q such that $Q = \mathcal{G}(V)$ and $\mathcal{Q} = \langle Q\rangle \subset F^r_+$.*

In the following, given a reduced F-matrix V, we always assume the cone $\mathcal{Q} \subseteq F^r_+$ and generated by the columns of a positive REF matrix $Q = \mathcal{G}(V)$.

Definition 1.7. Let $\mathcal{S}_r$ be the family of all r-dimensional subcones of $\mathcal{Q}$ obtained as intersection of simplicial subcones of $\mathcal{Q}$. Then define the *secondary fan* (or *GKZ decomposition*) of V to be the set $\Gamma = \Gamma(V)$ of cones in F^r_+ such that

- its subset of r-dimensional cones (the r-*skeleton*) $\Gamma(r)$ is composed by the minimal elements, with respect to the inclusion, of the family $\mathcal{S}_r$,
- its subset of i-dimensional cones (the i-*skeleton*) $\Gamma(i)$ is composed by all the i-dimensional faces of cones in $\Gamma(r)$, for $1 \le i \le r-1$.

A maximal cone $\gamma \in \Gamma(r)$ is called a *chamber* of the secondary fan Γ. Finally define

$$\text{Mov}(V) := \bigcap_{i=1}^{n+r} \langle Q^{\{i\}}\rangle, \tag{1}$$

where $\langle Q^{\{i\}}\rangle$ is the cone generated in F^r_+ by the columns of the submatrix $Q^{\{i\}}$ of Q (see the list of notation 1.1).

Theorem 1.8. *If V is an F-matrix then, for every $\Sigma \in \mathcal{SF}(V)$,*

(1) $\mathcal{Q} = \overline{\mathrm{Eff}}(X(\Sigma))$, *the* pseudo-effective cone of X, *which is the closure of the cone generated by effective Cartier divisor classes of X, see* [9, Lemma 15.1.8],

(2) $\mathrm{Mov}(V) = \overline{\mathrm{Mov}}(X(\Sigma))$, *the closure of the cone generated by movable Cartier divisor classes of X, see* [9, (15.1.5), (15.1.7), Theorem 15.1.10, Proposition 15.2.4].

(3) *$\Gamma(V)$ is the secondary fan (or GKZ decomposition) of $X(\Sigma)$, see* [9, § 15.2]. *In particular Γ is a fan and* $|\Gamma| = \mathcal{Q} \subset F^r_+$.

Theorem 1.9 ([9] Proposition 15.2.1). *There exists a one to one correspondence between the two sets $\mathcal{A}_\Gamma(V) := \{\gamma \in \Gamma(r) \mid \gamma \subset \mathrm{Mov}(V)\}$ and $\mathbb{P}\mathcal{SF}(V) := \{\Sigma \in \mathcal{SF}(V) \mid X(\Sigma)$ is projective$\}$.*

For the following it is useful to understand the construction of such a correspondence (compare [9] Proposition 15.2.1). After [4], given a chamber $\gamma \in \mathcal{A}_\Gamma$ we call *the bunch of cones of γ* the collection of cones in F^r_+ given by

$$\mathcal{B}(\gamma) := \{\langle Q_J\rangle \mid J \subset \{1,\dots,n+r\}, |J| = r, \det(Q_J) \neq 0, \gamma \subset \langle Q_J\rangle\}$$

(see also [9, p. 738]). It turns out that $\bigcap_{\beta\in\mathcal{B}(\gamma)} \beta = \gamma$, and that for any $\gamma \in \mathcal{A}_\Gamma(V)$ there exists a unique fan $\Sigma_\gamma \in \mathbb{P}\mathcal{SF}(V)$ such that

$$\Sigma_\gamma(n) := \{\langle V^J\rangle \mid \langle Q_J\rangle \in \mathcal{B}(\gamma)t\}.$$

For any $\Sigma \in \mathbb{P}\mathcal{SF}(V)$ the collection of cones

$$\mathcal{B}_\Sigma := \{\langle Q^I\rangle \mid \langle V_I\rangle \in \Sigma(n)\} \tag{2}$$

is the bunch of cones of the chamber $\gamma_\Sigma \in \mathcal{A}_\Gamma$ given by $\gamma_\Sigma := \bigcap_{\beta\in\mathcal{B}_\Sigma} \beta$. The correspondence $\mathcal{A}_\Gamma(V) \leftrightarrow \mathbb{P}\mathcal{SF}(V)$ in Theorem 1.9 is realized by $\gamma \mapsto \Sigma_\gamma$ and $\gamma_\Sigma \leftarrow\!\shortmid \Sigma$.

Remark 1.10. Note that in the previous picture we get well established bijections

$$\text{for all } \gamma \in \mathcal{A}_\Gamma(V) \quad \begin{array}{ccc} \mathcal{B}(\gamma) & \longleftrightarrow & \Sigma_\gamma(n) \\ \langle Q_J\rangle & \longmapsto & \langle V^J\rangle \end{array} \qquad \text{and} \qquad \text{for all } \Sigma \in \mathbb{P}\mathcal{SF}(V) \quad \begin{array}{ccc} \Sigma(n) & \longleftrightarrow & \mathcal{B}_\Sigma \\ \langle V_I\rangle & \longmapsto & \langle Q^I\rangle \end{array}.$$

A significant consequence of [22, Corollary 3.3] is that these bijections preserve, possibly up to a constant integer, the determinants of generating submatrices, which means that

$$\delta\cdot|\det(Q_J)| = |\det(V^J)| \quad \text{and} \quad |\det(V_I)| = \delta\cdot|\det(Q^I)|, \tag{3}$$

where $\delta = 1$ if and only if V is a CF-matrix. Therefore, in the following, when V is a CF-matrix, a chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is called *non-singular* if the bunch of cones $\mathcal{B}(\gamma)$ is entirely composed of unimodular cones (as defined in list 1.1) or, equivalently, if the associated fan $\Sigma_\gamma \in \mathbb{P}\mathcal{SF}(V)$ is non-singular.

As a final result we recall the following

Proposition 1.11 ([9] Theorem 15.1.10(c)). *If $V = (\mathbf{v}_1 \dots, \mathbf{v}_{n+r})$ is an F-matrix then, for every fan $\Sigma \in \mathbb{P}\mathcal{SF}(V)$, there is a natural isomorphism* $\mathrm{Pic}(X(\Sigma)) \otimes \mathbb{R} \cong F^r_{\mathbb{R}}$ *taking the cones*

$$\mathrm{Nef}(X(\Sigma)) \subseteq \overline{\mathrm{Mov}}(X(\Sigma)) \subseteq \overline{\mathrm{Eff}}(X(\Sigma))$$

to the cones $\gamma_\Sigma \subseteq \mathrm{Mov}(V) \subseteq \mathcal{Q}$. In particular, calling $d : \mathcal{W}_T(X(\Sigma)) \to \mathrm{Cl}(X(\Sigma))$ the morphism assigning to a torus invariant divisor D its linear equivalence class $d(D)$, we obtain the following:

(1) *a $\mathbb{Q}$-Cartier divisor D on $X(\Sigma)$ is a nef (ample) divisor if and only if $d(D) \in \gamma_\Sigma$ ($d(D) \in \mathrm{Relint}(\gamma_\Sigma t)$, where* Relint *denotes the interior of the cone γ_Σ in its linear span.*

(2) *$X(\Sigma)$ is $\mathbb{Q}$-Fano if and only if*

$$\sum_{j=1}^{n+r} d(D_j) \in \mathrm{Relint}(\gamma_\Sigma)$$

where D_j is the closure of the torus orbit of the ray $\langle\mathbf{v}_j\rangle$.

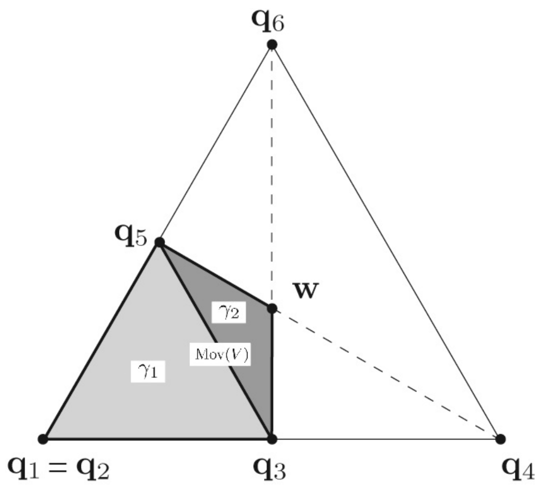

Figure 1: Example 2.40: the section of the cone Mov(V) and its chambers, inside the Gale dual cone $\mathcal{Q} = F_+^3$, as cut out by the plane $\sum_{i=1}^3 x_i^2 = 1$.

Example 1.12 (Example 1.5 continued). Let $X(\Sigma)$ be one of the two smooth and projective toric varieties defined in Example 1.5. One can visualize the pseudo-effective cone $\overline{\mathrm{Eff}}(X(\Sigma))$, that is the Gale dual cone $\mathcal{Q} = \langle Q \rangle = F_+^3$, and the movable cone $\mathrm{Mov}(V) \subseteq \mathcal{Q}$ by giving a picture of their section with the hyperplane $\{x_1 + x_2 + x_3 = 1\} \subseteq F_{\mathbb{R}}^3 \cong \mathrm{Cl}(X) \otimes \mathbb{R}$, as in Figure 1. Then $\mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V)$ can be dually described by the only two chambers of Mov(V) represented in Figure 1 and explicitly given by

$$\gamma_1 = \langle \mathbf{q}_1 = \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_5 \rangle = \left\langle \begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right\rangle, \quad \gamma_2 = \langle \mathbf{q}_3, \mathbf{w}, \mathbf{q}_5 \rangle = \left\langle \begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix} \right\rangle, \quad \mathrm{Mov}(V) = \langle \mathbf{q}_2, \mathbf{q}_3, \mathbf{w}, \mathbf{q}_5 \rangle = \gamma_1 + \gamma_2$$

where, as usual, $\mathbf{q}_1, \dots, \mathbf{q}_6$ are the columns of Q and $\mathbf{w} := \mathbf{q}_3 + \mathbf{q}_6 = \mathbf{q}_4 + \mathbf{q}_5$. Then we have $\mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V) = \{\Sigma_1 = \Sigma_{\gamma_1}, \Sigma_2 = \Sigma_{\gamma_2}\}$ and $\gamma_i = \mathrm{Nef}(X(\Sigma_i))$ for $i = 1, 2$.

1.3 Toric flips. In the present context, a *toric flip* will be a torus-equivariant birational equivalence of projective $\mathbb{Q}$-factorial toric varieties which is an isomorphism in codimension 1. A toric flip is a composition of *elementary flips* and a toric isomorphism, see [9, Theorem 15.3.14]: given a reduced F-matrix V, an elementary flip is defined as the birational equivalence realized by passing, inside Mov(V), from a chamber to another one, just *crossing a wall* [9, (15.3.14)].

E.g. in the previous Examples 1.5 and 1.12, the smooth projective toric varieties $X(\Sigma_1)$ and $X(\Sigma_2)$ are related by an elementary flip, obtained by crossing the wall determined by cutting $\mathcal{Q}$ with the plane containing $\mathbf{q}_3$ and $\mathbf{q}_5$ (see Figure 1). Hence they are isomorphic in codimension 1.

2 The Batyrev classification revised

In this section we propose an alternative approach to Batyrev's results presented in [2, § 3, 4], not depending on the smoothness hypothesis and holding for the case of a $\mathbb{Q}$-factorial projective toric variety.

2.1 Primitive relations and bordering chambers. Given a reduced F-matrix $V = (\mathbf{v}_1, \dots, \mathbf{v}_{n+r})$ and a fan $\Sigma \in \mathcal{SF}(V)$, the datum of a collection of rays $\mathcal{P} = \{\rho_1, \dots, \rho_k\} \subseteq \Sigma(1)$ determines a subset $P = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n+r\}$ such that $\mathcal{P} = \{\rho_1, \dots, \rho_k\} = \{\langle \mathbf{v}_{j_1} \rangle, \dots, \langle \mathbf{v}_{j_k} \rangle\}$ and a submatrix V_P of V.

2.1.1 Notation. By abuse of notation we will often write

$$\mathcal{P} = \{\mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_k}\} = \{V_P\}.$$

From the point of view of the Gale dual cone $\mathcal{Q} = \langle Q \rangle$, where $Q = (\mathbf{q}_1, \dots, \mathbf{q}_{n+r}) = \mathcal{G}(V)$ is a reduced, positive, REF, W-matrix, the subset $P \subseteq \{1, \dots, n+r\}$ determines the collection $\mathcal{P}^* = \{\langle \mathbf{q}_{j_1} \rangle, \dots, \langle \mathbf{q}_{j_k} \rangle\} \subseteq \Gamma(1)$. By the same abuse of notation we will often write

$$\mathcal{P}^* = \{\mathbf{q}_{j_1}, \ldots, \mathbf{q}_{j_k}\} = \{Q_P\}.$$

The vector $\mathbf{v}_\mathcal{P} := \sum_{i=1}^k \mathbf{v}_{j_i}$ lies in the relative interior of a cone $\sigma \in \Sigma$ and there is a unique relation

$$\mathbf{v}_\mathcal{P} - \sum_{\rho\in\sigma(1)} c_\rho \mathbf{v}_\rho = 0 \quad \text{with } \langle \mathbf{v}_\rho \rangle = \rho \cap N \text{ and } c_\rho \in \mathbb{Q}, c_\rho > 0. \tag{4}$$

This fact allows us to define a rational vector $r(P) = r(\mathcal{P}) = (b_1, \ldots, b_{n+r}) \in \mathbb{Q}^{n+r}$, where b_j is the coefficient of the column $\mathbf{v}_j$ of V in (4). Let l be the least common denominator of $b_1, \ldots, b_{n+r}$. Then

$$r_\mathbb{Z}(P) = r_\mathbb{Z}(\mathcal{P}) := lr(\mathcal{P}) = (lb_1, \ldots, lb_{n+r}) \in \mathcal{L}_r(Q) \subset \mathbb{Z}^{n+r}. \tag{5}$$

Recall that a collection $\mathcal{P} \subset \Sigma(1)$ is called *primitive for Σ* if it is not contained in a single cone of Σ but every proper subset of $\mathcal{P}$ is; compare [2, Definition 2.6], [10, Definition 1.1], [9, Definition 5.1.5]. If $\mathcal{P}$ is a primitive collection then it is determined by the positive entries in $r_\mathbb{Z}(\mathcal{P})$ (for the details see [10, Lemma 1.8]); this is no more the case if $\mathcal{P}$ is not a primitive collection.

Consider the $\mathbb{Q}$-factorial complete toric variety $X = X(\Sigma)$; the standard exact sequence on divisors is

$$0 \longrightarrow M \xrightarrow[V^T]{div} \mathcal{W}_T(X) = \mathbb{Z}^{n+r} \xrightarrow{d} \mathrm{Cl}(X) \longrightarrow 0\ , \tag{6}$$

where $\mathcal{W}_T(X)$ denotes the group of torus-invariant Weil divisors. Dualizing this sequence, one gets the following exact sequence of free abelian groups

$$0 \longrightarrow A_1(X) := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Z}) \xrightarrow[Q^T]{d^\vee} \mathrm{Hom}(\mathcal{W}_T(X), \mathbb{Z}) = \mathbb{Z}^{n+r} \xrightarrow[V]{div^\vee} N \tag{7}$$

Then (5) gives that $r_\mathbb{Z}(P) \in \mathrm{Im}(d^\vee)$. Since $d^\vee$ is injective there exists a unique $\mathbf{n}_P \in A_1(X)$ such that

$$d^\vee(\mathbf{n}_P) = Q^T \cdot \mathbf{n}_P = r_\mathbb{Z}(P) \tag{8}$$

which turns out to be the numerical equivalence class of the 1-cycle $r_\mathbb{Z}(P)$, whose intersection index with the torus-invariant Weil divisor lD_j is given by the integer lb_j, for $1 \le j \le n + r$. In particular, given a primitive collection $\mathcal{P}$ the associated primitive relation $r_\mathbb{Z}(P)$ is a numerically effective 1-cycle (nef) if and only if all the coefficients lb_j in (5) are non-negative: in this case $\mathcal{P}$ is called *a numerically effective (nef) primitive collection*.

Definition 2.1. Given a collection $\mathcal{P} = \{V_P\}$, for $P \subseteq \{1, \ldots, n+r\}$, its associated numerical class $\mathbf{n}_P \in N_1(X) := A_1(X) \otimes \mathbb{R}$, defined in (8), determines a unique dual hyperplane

$$H_P \subseteq F^r_\mathbb{R} = \mathrm{Cl}(X) \otimes \mathbb{R}$$

which is called *the support* of $\mathcal{P}$, a *positive half-space* $\mathcal{H}^+_P := \{\mathbf{x} \in F^r_\mathbb{R} \mid \mathbf{n}_P \cdot \mathbf{x} \ge 0\}$ and a *negative half-space* $\mathcal{H}^-_P := \{\mathbf{x} \in F^r_\mathbb{R} \mid \mathbf{n}_P \cdot \mathbf{x} \le 0\}$.

Denoting by $\mathfrak{P} = \mathfrak{P}(\{1, \ldots, n + r\})$ the power set of $\{1, \ldots, n + r\}$, the otation introduced in 2.1.1 allows us to think of the set of primitive collections of a fan $\Sigma \in \mathcal{SF}(V)$ as a suitable subset of $\mathfrak{P}$, namely

$$\mathrm{PC}(\Sigma) = \{P \in \mathfrak{P} \mid \{V_P\} \text{ is a primitive collection}\}.$$

The following proposition gives some further characterization of primitive collections.

Proposition 2.2. *Let V be a reduced F-matrix, $Q = \mathcal{G}(V)$ a Gale dual* REF, *positive W-matrix, $\Sigma \in \mathbb{P}\mathcal{SF}(V)$ and $P \in \mathrm{PC}(\Sigma)$ such that $\mathcal{P} = \{V_P\}$ is a primitive collection for Σ. Then $|\mathcal{P}| = |P| \le n + 1$ and the following are equivalent:*

(1) *$\mathcal{P}$ is a primitive collection for Σ, which is*
   (i.1) $\forall \sigma \in \Sigma(n) : \mathcal{P} \nsubseteq \sigma(1)$,
   (ii.1) $\forall \rho_i \in \mathcal{P}\ \exists \sigma \in \Sigma(n) : \mathcal{P}\backslash\{\rho_i\} \subseteq \sigma(1)$;

(2) *V_P is a submatrix of V such that*
    (i.2) $\forall J \subseteq \{1, \dots, n+r\} : \langle V_J \rangle \in \Sigma(n),\ \langle V_P \rangle \nsubseteq \langle V_J \rangle$,
    (ii.2) $\forall i \in P\ \exists J \subseteq \{1, \dots, n+r\} : \langle V_J \rangle \in \Sigma(n),\ \langle V_{P\setminus\{i\}} \rangle \subseteq \langle V_J \rangle$;
(3) *Q_P is a submatrix of $Q = \mathcal{G}(V)$ such that*
    (i.3) $\forall J \subseteq \{1, \dots, n+r\} : \langle Q^J \rangle \in \mathcal{B}(\gamma_\Sigma),\ \langle Q^J \rangle \nsubseteq \langle Q^P \rangle$,
    (ii.3) $\forall i \in P\ \exists J \subseteq \{1, \dots, n+r\} : \langle Q^J \rangle \in \mathcal{B}(\gamma_\Sigma),\ \langle Q^J \rangle \subseteq \langle Q^{P\setminus\{i\}} \rangle$;
(4) *Q_P is a submatrix of $Q = \mathcal{G}(V)$ such that*
    (i.4) $\gamma_\Sigma \nsubseteq \langle Q^P \rangle$,
    (ii.4) $\forall i \in P : \gamma_\Sigma \subseteq \langle Q^{P\setminus\{i\}} \rangle$.

Moreover the previous conditions (ii.1), (ii.2), (ii.3), (ii.4) *are equivalent to the following one:*

(ii) $\forall i \in P\ \exists \mathcal{C}_{i,P} \in \mathcal{B}(\gamma_\Sigma) : \mathcal{C}_{i,P}(1) \cap \mathcal{P}^* = \{\langle \mathbf{q}_i \rangle\}$.

Proof. Note that if $|\mathcal{P}| \geq n+2$ then condition (ii.1) cannot be satisfied, since every cone $\sigma \in \Sigma(n)$ is simplicial, implying that $|\sigma(1)| = n < n+1 \leq |\mathcal{P}| - 1$. Then $|\mathcal{P}| \leq n+1$ for a primitive collection.

The equivalence (1) $\Leftrightarrow$ (2) is clear. The equivalence (2) $\Leftrightarrow$ (3) follows by Gale duality and Theorem 1.9. We consider the equivalence (3) $\Leftrightarrow$ (4).

(i.3) $\Rightarrow$ (i.4): (i.4) is always true when $|P| = n+1$ because $\dim(\langle Q^P \rangle) \leq r-1$. Let us then assume that $|P| \leq n$ and $\gamma_\Sigma \subseteq \langle Q^P \rangle$. Then there certainly exists a simplicial subcone of $\langle Q^P \rangle$ containing γ_Σ, which is

$$\exists J \subseteq \{1, \dots, n+r\} : \langle Q^J \rangle \in \mathcal{B}(\gamma_\Sigma), \langle Q^J \rangle \subseteq \langle Q^P \rangle \tag{9}$$

contradicting (i.3).

(i.4) $\Rightarrow$ (i.3): Assume (9). Then $\gamma_\Sigma \subseteq \langle Q^J \rangle \subseteq \langle Q^P \rangle$, contradicting (i.4).

(ii.3) $\Rightarrow$ (ii.4): By (ii.3), $\gamma_\Sigma \subseteq \langle Q^J \rangle \subseteq \langle Q^{P\setminus\{i\}} \rangle$, clearly giving (ii.4).

(ii.4) $\Rightarrow$ (ii.3): Since $|P| - 1 \leq n$, assuming $\gamma_\Sigma \subseteq \langle Q^{P\setminus\{i\}} \rangle$ always gives a simplicial subcone $\langle Q^J \rangle$ of $\langle Q^{P\setminus\{i\}} \rangle$ containing γ_Σ. Then (ii.3) follows.

For the last part:

(ii.4) $\Rightarrow$ (ii): if $|P| - 1 = n$ then define $\mathcal{C}_{i,P} := \langle Q^{P\setminus\{i\}} \rangle$ which is a simplicial cone; if $|P| \leq n$ then $\langle Q^{P\setminus\{i\}} \rangle \supseteq \gamma_\Sigma$ and $|\langle Q^{P\setminus\{i\}} \rangle(1)| \geq r+1$; consider the simplicial star-subdivision of $\langle Q^{P\setminus\{i\}} \rangle$ having center in the ray $\mathbf{q}_i \in \mathcal{P}^*$; in this subdivision let $\mathcal{C}_{i,P}$ be the unique simplicial subcone containing γ_Σ, which exists by the definition of the secondary fan Γ; since $\mathcal{C}_{i,P} \subseteq \langle Q^{P\setminus\{i\}} \rangle$ we have $\mathcal{C}_{i,P}(1) \cap (\mathcal{P}^* \setminus \{\langle \mathbf{q}_i \rangle\}) = \emptyset$; but $\langle \mathbf{q}_i \rangle \in \mathcal{C}_{i,P}(1)$ by construction; then $\mathcal{C}_{i,P}(1) \cap \mathcal{P}^* = \{\langle \mathbf{q}_i \rangle\}$;

(ii) $\Rightarrow$ (ii.3): set $\langle Q^J \rangle := \mathcal{C}_{i,P}$; then $\langle Q^J \rangle(1) \cap (\mathcal{P}^* \setminus \{\langle \mathbf{q}_i \rangle\}) = \mathcal{C}_{i,P}(1) \cap (\mathcal{P}^* \setminus \{\langle \mathbf{q}_i \rangle\}) = \emptyset$, hence $\langle Q^J \rangle \subseteq \langle Q^{P\setminus\{i\}} \rangle$. □

Let $\overline{\mathrm{NE}}(X) \subseteq N_1(X)$ be the Mori cone, generated by the numerical classes of effective curves. Reid proved that $\overline{\mathrm{NE}}(X)$ is closed and polyhedral when X is a $\mathbb{Q}$-factorial complete toric variety (see [20, Corollary (1.7)]) and generated by classes of torus-invariant curves. When X is smooth, Casagrande ensures that the numerical class $\mathbf{n}_P$, of a primitive relation $r_{\mathbb{Z}}(P)$, belong to $A_1(X) \cap \overline{\mathrm{NE}}(X)$; see [5, Lemma 1.4]. Cox and von Renesse [10, Propositions 1.9 and 1.10] generalize and improve this fact to toric varieties whose fan has convex support, showing that the Mori cone is generated by numerical classes of primitive relations, namely

$$\overline{\mathrm{NE}}(X) = \sum_{P \in \mathrm{PC}(\Sigma)} \mathbb{R}_+ \mathbf{n}_P. \tag{10}$$

In particular we get the following

Proposition 2.3 (Lemma 1.4 in [5], Proposition 1.9 in [10]). *If $\mathcal{P} = \{V_P\}$ is a primitive collection, for some $P \in \mathrm{PC}(\Sigma)$, then its numerical class $\mathbf{n}_P$ is positive against every nef divisor of $X(\Sigma)$, which is $\gamma_\Sigma \subseteq \mathcal{H}_P^+$.*

Dualizing (10) we get the following description of the closure of the Kähler cone:

$$\mathrm{Nef}(X) = \bigcap_{P \in \mathrm{PC}(\Sigma)} \mathcal{H}_P^+. \tag{11}$$

Then Proposition 2.2 allows us to give the following alternative description of this cone:

Corollary 2.4. *Let V be a reduced F-matrix, $Q = \mathcal{G}(V)$ be a* REF, *positive W-matrix and $\Sigma \in \mathbb{PSF}(V)$. Then*

$$\operatorname{Nef}(X(\Sigma)) = \bigcap_{i\in P\in \mathrm{PC}(\Sigma)} \langle Q^{P\setminus\{i\}}\rangle .$$

Proof. By Proposition 1.11, $\operatorname{Nef}(X(\Sigma)) = \gamma_\Sigma = \bigcap_{\beta\in\mathcal{B}(\gamma_\Sigma)} \beta$; then clearly

$$\operatorname{Nef}(X(\Sigma)) \subseteq \bigcap_{i\in P\in \mathrm{PC}(\Sigma)} \mathcal{C}_{i,P} \subseteq \bigcap_{i\in P\in \mathrm{PC}(\Sigma)} \langle Q^{P\setminus\{i\}}\rangle$$

where $\mathcal{C}_{i,P} \in \mathcal{B}(\gamma_\Sigma)$ are the cones defined in condition (ii) of Proposition 2.2. For the converse note that

$$\forall\, P \in \mathrm{PC}(\Sigma) : \langle Q^{P\setminus\{i\}}\rangle = \langle Q^P\rangle \cup (\mathcal{H}_P^+ \cap \langle Q^{P\setminus\{i\}}\rangle).$$

But $\langle Q^P\rangle \subseteq \mathcal{H}_P^-$ and $\Sigma \in \mathbb{PSF}(V)$ implies that $\dim(\operatorname{Nef}(X)) = \dim(\gamma_\Sigma) = r$. Then (11) gives

$$\bigcap_{i\in P\in \mathrm{PC}(\Sigma)} \langle Q^{P\setminus\{i\}}\rangle = \bigcap_{i\in P\in \mathrm{PC}(\Sigma)} (\mathcal{H}_P^+ \cap \langle Q^{P\setminus\{i\}}\rangle) \subseteq \bigcap_{P\in \mathrm{PC}(\Sigma)} \mathcal{H}_P^+ = \operatorname{Nef}(X). \qquad \square$$

Definition 2.5 (Bordering collections and chambers).

(1) Let V be a reduced F-matrix and $Q = \mathcal{G}(V)$ a REF, positive W-matrix. A collection $\mathcal{P} = \{V_P\}$, for some $P \in \mathfrak{P}$, is called *bordering* if its support H_P cuts out a facet of the Gale dual cone $\mathcal{Q} = \langle Q\rangle$.

(2) A chamber $\gamma \in \Gamma(V)$ is called *bordering* if $\dim(\gamma \cap \partial\mathcal{Q}) \geq 1$. Note that $\gamma \cap \partial\mathcal{Q}$ is always composed of faces of γ: if it contains a facet of γ then γ is called *maximally bordering* (*maxbord* for short). A hyperplane H cutting a facet of $\mathcal{Q}$ and such that $\dim(\gamma\cap H) \geq 1$ is called a *bordering hyperplane of* γ, and the bordering chamber γ is also called *bordering with respect to* H. A normal vector $\mathbf{n}$ to a bordering hyperplane H is called *inward* if $\mathbf{n}\cdot x \geq 0$ for every $x \in \gamma$.

Remark 2.6. We give a geometric interpretation of concepts introduced in the previous Definition 2.5.

(1) $P \in \mathrm{PC}(\Sigma)$ gives a bordering primitive collection $\mathcal{P} = \{V_P\}$ if and only $\mathcal{P}$ is nef, which means that $r_{\mathbb{Z}}(\mathcal{P})$ is a numerically effective 1-cycle.

(2) Thinking of $\gamma \in \Gamma(V)$ as the cone $\operatorname{Nef}(X(\Sigma_\gamma)) \subseteq \operatorname{Eff}(X(\Sigma_\gamma))$, the chamber γ turns out to be bordering if and only if $X(\Sigma_\gamma)$ admits non-trivial effective divisors which are nef but non-big; see [12]. Following [15] and [6], this is equivalent to the existence of a *rational contraction of fiber type* $f : X \dashrightarrow Y$ to a normal projective toric variety Y.

Remark 2.7. Let $\gamma \in \Gamma(V)$ be a bordering chamber and let H be a bordering hyperplane of γ. Then there exist at least $r-1$ columns of $Q = \mathcal{G}(V)$ belonging to H. Let $\mathcal{C}_H$ be the $(r-1)$-dimensional cone generated by all the columns of Q belonging to H. Then

$$\gamma \cap H \subset \mathcal{C}_H. \tag{12}$$

In fact $\mathcal{Q} = |\Gamma(V)|$ and $\gamma \subset \mathcal{Q}$, giving that $\gamma \cap H \subset \mathcal{Q}\cap H = \mathcal{C}_H$.

Proposition 2.8. *Let V be a reduced F-matrix, $Q = \mathcal{G}(V)$ a positive,* REF *W-matrix and $\gamma \in \Gamma(r) \subseteq \mathcal{Q} = \langle Q\rangle$. Then γ is a maxbord chamber with respect to a hyperplane H if and only if*

$$\forall\, \beta \in \mathcal{B}(\gamma)\ \exists \mathbf{q} \in \mathcal{Q}(1)\backslash\mathcal{C}_H(1) : \beta = \langle \mathbf{q}\rangle + \beta \cap H,$$

where $\mathcal{C}_H$ is the $(r-1)$-dimensional cone generated by all the columns of Q belonging to H.

Proof. If γ is maxbord with respect to H then $\dim(\gamma\cap H) = r-1$ implies that $\dim(\beta\cap H) = r-1$ for all $\beta \in \mathcal{B}(\gamma)$. Since β is simplicial, this suffices to show that there exists a unique $\mathbf{q} \in \beta(1)$ not belonging to H and such that $\beta = \langle\mathbf{q}\rangle + \beta\cap H$.

The converse follows immediately by recalling that $\gamma = \bigcap_{\beta\in\mathcal{B}(\gamma)} \beta$ and we are assuming $\dim(\gamma) = r$. $\square$

Definition 2.9. Let V be a reduced F-matrix. A bordering chamber $\gamma \in \Gamma(V)$, with respect to the hyperplane H, is called *internal bordering* (*intbord* for short) with respect to H, if either γ is maxbord with respect to H or there exists an hyperplane H', cutting a facet of γ and such that

(i) $\gamma \cap H \subseteq \gamma \cap H'$

(ii) $\exists\, \mathbf{q}_1, \mathbf{q}_2 \in H \cap \mathcal{Q}(1) : (\mathbf{n}' \cdot \mathbf{q}_1)(\mathbf{n}' \cdot \mathbf{q}_2) < 0$

where $\mathbf{n}'$ is the inward primitive normal vector of H'.

Remark 2.10. For Picard number $r \leq 2$, a chamber $\gamma \in \Gamma(V)$ is bordering with respect to a hyperplane H if and only if it is intbord with respect to H if and only if it is maxbord with respect to H. For $r \geq 3$, this is no more the case but

$$\text{maxbord with respect to } H \implies \text{intbord with respect to } H \implies \text{bordering with respect to } H$$

and there exist chambers which are either bordering and not intbord or intbord and not maxbord with respect to H.

The following result gives the existence of a bordering primitive collection, hence of a numerically effective primitive relation, for a $\mathbb{Q}$-factorial projective toric variety whose fan corresponds to an intbord chamber of the secondary fan $\Gamma(V)$. This is one of the key results of the present paper, allowing us to improve and extend the Batyrev classification, explained in [2], to the case of singular $\mathbb{Q}$-factorial projective toric varieties. In some sense, the following result is the analogue, in a singular setup, of Batyrev's result [2, Proposition 3.2] (see the following Remark 2.15).

Theorem 2.11. *Let V be a reduced F-matrix and let $\gamma \in \mathcal{A}_\Gamma(V)$ be a bordering chamber with respect to a hyperplane H. Then γ is an intbord chamber with respect to H if and only if the hyperplane H is the support of a bordering primitive collection $\mathcal{P}$, for the fan $\Sigma_\gamma \in \mathbb{P}\mathcal{SF}(V)$.*

Proof. Let H be a bordering hyperplane for γ. Let us assume, up to a permutation of columns of $Q = \mathcal{G}(V)$, that the first $s \geq r-1$ columns $\mathbf{q}_1, \dots, \mathbf{q}_s$ are all the columns of Q belonging to H. Setting $P = \{s+1, \dots, n+r\} \in \mathfrak{P}$, consider the collection $\mathcal{P} = \{V_P\}$. We want to show that $\mathcal{P}$ is a primitive collection for Σ_γ. On the one hand, condition (i.4) in Proposition 2.2 is immediately satisfied since $\det(Q^P) = \det(Q_{\{1,\dots,s\}}) = 0$, as $\mathbf{q}_1, \dots, \mathbf{q}_s \in H$. On the other hand, to show that condition (ii.4) in Proposition 2.2 holds, note that for every $i \in P$ condition (ii) in Definition 2.9 ensures that $\gamma \subseteq \langle \mathbf{q}_i \rangle + \mathcal{C}_H = \langle Q^{P\setminus\{i\}} \rangle$ where $\mathcal{C}_H = H \cap \mathcal{Q}$, as defined in Remark 2.7.

For the converse, let $\mathcal{P} = \{V_P\}$ be a bordering primitive collection with respect to the hyperplane H. By (11), H turns out to cut out a face of $\mathrm{Nef}(X)$. If H cut out a facet of $\mathrm{Nef}(X) = \gamma$ then γ turns out to be maxbord with respect to H, hence intbord. Let us then assume that $\dim(H \cap \mathrm{Nef}(X)) \leq r-2$. This means that the numerical class $\mathbf{n}_P$ is not extremal in the decomposition (10) of the Mori cone. Then there exist $l \geq 2$ extremal classes $\mathbf{n}_1, \dots, \mathbf{n}_l \in \partial\overline{\mathrm{NE}}(X)$ such that $\mathbf{n}_P = \sum_{k=2}^{l} \mu_k \mathbf{n}_k$, for some $\mu_k > 0$. Let $H_k \subseteq F^r_{\mathbb{R}}$ be the dual hyperplane to $\mathbf{n}_k$, for $1 \leq k \leq l$, which by construction cuts out a facet of γ. Since we are assuming γ to be bordering with respect to H, we have $\gamma \cap H = \gamma \cap \left(\bigcap_{k=2}^{l} H_k\right)$. Hence, by Definition 2.9, if γ would not be intbord with respect to H then

$$\forall\, k = 1, \dots, l: \text{ either } \forall\, j \notin P \quad \mathbf{n}_k \cdot \mathbf{q}_j \leq 0 \quad \text{or} \quad \forall\, j \notin P \quad \mathbf{n}_k \cdot \mathbf{q}_j \geq 0.$$

In particular, since $H_k \neq H$, there exists $j \notin P$ such that $\mathbf{n}_k \cdot \mathbf{q}_j \neq 0$. But $\mathbf{q}_j \in H$, giving

$$\begin{aligned} 0 = \mathbf{n}_P \cdot \mathbf{q}_j = \sum_{k=2}^{l} \mu_k \mathbf{n}_k \cdot \mathbf{q}_j &\implies \exists\, 1 \leq k_0 \leq l : \mathbf{n}_{k_0} \cdot \mathbf{q}_{j_0} < 0 \\ &\implies \forall\, j \notin P\ \mathbf{n}_{k_0} \cdot \mathbf{q}_j \leq 0. \end{aligned}$$

By construction there exists $i \in P$ such that $\mathbf{q}_i \in H_{k_0} \cap \mathcal{P}$. Then $\langle Q^{P\setminus\{i\}} \rangle = \langle \mathbf{q}_i \rangle + \mathcal{C}_H$. On the one hand $\gamma \subseteq \langle Q^{P\setminus\{i\}} \rangle$, by condition (ii.4) of Proposition 2.2. Therefore there exists $\mathbf{x} \in \gamma \subseteq \langle Q^{P\setminus\{i\}} \rangle$ such that $\mathbf{n}_{k_0} \cdot \mathbf{x} > 0$, where $\mathbf{n}_{k_0}$ is the primitive inward normal vector to the facet H_{k_0} of γ. On the other hand

$$\forall\, \mathbf{x} \in \langle Q^{P\setminus\{i\}} \rangle = \langle \mathbf{q}_i \rangle + \mathcal{C}_H \ : \ \mathbf{n}_{k_0} \cdot \mathbf{x} = \lambda_i \mathbf{n}_{k_0} \cdot \mathbf{q}_i + \sum_{j \notin P} \lambda_j \mathbf{n}_{k_0} \cdot \mathbf{q}_j = \sum_{j \notin P} \lambda_j \mathbf{n}_{k_0} \cdot \mathbf{q}_j \leq 0$$

giving a contradiction. Then γ has to be intbord with respect to H. □

2.1.2 Notation. Calling $x_1, \dots, x_r$ the coordinates of $F^r_{\mathbb{R}} = \mathbb{R}^r$, in the following H_i denotes the coordinate hyperplane $x_i = 0$; in particular $H_r := \{x_r = 0\}$.

Corollary 2.12. *Let V be a reduced F-matrix and let $\gamma \in \mathcal{A}_\Gamma(V)$ be an intbord and non-singular chamber. Then the associated fan $\Sigma_\gamma \in \mathbb{P}\mathcal{SF}(V)$ admits a nef primitive collection $\mathcal{P}$ whose primitive relation* (5) *has all the non-zero coefficients equal to* 1.

Proof. This is immediate after [2, Proposition 3.2]. Alternatively, the previous Theorem 2.11 gives a bordering, hence nef, primitive collection $\mathcal{P} = \{V_P\} \subset \Sigma_\gamma(1)$ with $P = \{s+1, \dots, n+r\}$. By the following Lemma 2.14, one can always assume that the bordering hyperplane H_P is given by $H_r := \{x_r = 0\}$, recalling notation 2.1.2, and that Q is a positive, REF, W-matrix. By (8), this means that

$$r_{\mathbb{Z}}(P) = Q^T \cdot \mathbf{n}_P = Q^T \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = (0, \dots, 0, q_{r,s+1}, \dots, q_{r,n+r}),$$

i.e. the bottom row of Q gives the primitive relation $r_{\mathbb{Z}}(P)$. Then condition (ii) of Proposition 2.2 implies that, for $s+1 \leq i \leq n+r$, the column $\mathbf{q}_i$ of the weight matrix Q is always a generator of the simplicial cone $\mathcal{C}_{i,P} \in \mathcal{B}(\gamma)$, whose determinant is necessarily a multiple of the entry $q_{r,i}$ of Q. The non-singularity of γ then imposes $q_{r,i} = 1$, for all $i \in P$. □

Example 2.13 (Examples 1.5 and 1.12 continued). Consider the two isomorphic in codimension 1, smooth and projective toric varieties $X(\Sigma_1), X(\Sigma_2)$ defined in Example 1.5. Their chambers (i.e. Nef cones) γ_1, γ_2, respectively, are described in Example 1.12 and Figure 1. From the latter it is evident that both chambers are intbord with respect to both the hyperplanes H_2 and H_3, under notation 2.1.2, and moreover γ_1 is maxbord with respect to these hyperplanes. Theorem 2.11 then gives that H_2 and H_3 are supporting two collections, $\mathcal{P}_2 = \{\mathbf{v}_3, \mathbf{v}_4\}$ and $\mathcal{P}_3 = \{\mathbf{v}_5, \mathbf{v}_6\}$, respectively, which are primitive and nef for both the fans Σ_1 and Σ_2.

Lemma 2.14. *Let H be a hyperplane cutting a facet of $\mathcal{Q}$. Then there exist $\alpha \in \mathrm{GL}_r(\mathbb{Z})$ and a permutation matrix $\beta \in \mathrm{GL}_{n+r}(\mathbb{Z})$ such that $\alpha Q \beta$ is in* REF *and H is sent to the hyperplane H_r.*

Proof. Since H cuts a facet of $\mathcal{Q}$, up to a permutation of columns, one can assume that the first $s \geq r-1$ columns $\mathbf{q}_1, \dots, \mathbf{q}_s$ are all the columns of $Q = \mathcal{G}(V)$ belonging to H. Consider $\alpha' \in \mathrm{GL}_r(\mathbb{Z})$ and a permutation matrix $\beta' \in \mathrm{GL}_s(\mathbb{Z})$ such that $\alpha' Q_{\{1,\dots,s\}} \beta'$ is REF. Since there cannot exist r linearly independent vectors among $\mathbf{q}_1, \dots, \mathbf{q}_s$, the last r-th row of $\alpha' Q_{\{1,\dots,s\}} \beta'$ has to be $\mathbf{0}$, meaning that H has been sent to H_r. In particular the primitive inward normal vector of H has been transformed to $(0, \dots, 0, \pm 1)$. Therefore

$$\alpha' Q \begin{pmatrix} \beta' & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I}_{n+r-s} \end{pmatrix} = \left(\overbrace{\quad \mathrm{REF} \quad}^{s} \quad \overbrace{\begin{matrix} \vdots \\ q'_{r,s+1} \cdots q'_{r,n+r} \end{matrix}}^{n+r-s} \right)$$

with $q'_{r,s+1}, \dots, q'_{r,n+r}$ either strictly positive or strictly negative integer entries, depending on the sign of $(0, \dots, 0, \pm 1)$. The proof then concludes, possibly after a change of sign of the bottom r-th row, by adding suitable multiples of this latter row to the upper ones and by reordering the last $n+r-s$ columns to finally get a REF matrix. □

Remark 2.15. The previous Corollary 2.12 would give an alternative proof of Batyrev's result [2, Proposition 3.2] if it would be possible to prove that *a non-singular chamber is a bordering chamber*. Moreover Theorem 2.11 would give a generalization of this result of Batyrev to a singular setup. Actually this is the case for Picard number $r \leq 2$. In fact for $r = 1$ every chamber is maxbord. For $r = 2$, a non-singular chamber γ is maxbord, hence bordering. For $r = 3$, in [25] we prove that a non-singular chamber is bordering by assuming the existence of a nef primitive collection, hence assuming [2, Proposition 3.2]: this fact gives strong geometric consequences on smooth projective toric varieties of rank $r \leq 3$. Unfortunately for $r \geq 4$

non-singular chambers which are not bordering may exist: in fact, recalling Remark 2.6(2), in [12] Fujino and Sato exhibited examples of smooth projective toric varieties with $r \geq 5$ whose non-trivial nef divisors are big. We improved this result in [25, § 4.3, 4.4] to the case of Picard number $r = 4$.

For further comments, evidences and details, we refer the reader to [25].

2.2 Toric bundles and covers. This subsection introduces the main objects useful for the Batyrev-type classification in subsection 2.3.

2.2.1 Weighted Projective Toric Bundles (WPTB). We adopt an obvious generalization of notation and terminology given in [9] § 7.3 for a projective toric bundle (PTB).

Let $X'(\Sigma')$ be an n'-dimensional $\mathbb{Q}$-factorial complete toric variety, of rank r', and consider $s+1$ Cartier divisors $E_0, \ldots, E_s$ and the associated locally free sheaf $\mathcal{E} = \bigoplus_{k=0}^{s} \mathcal{O}_{X'}(E_k)$ of rank $s+1$. Let $W = (w_0, \ldots, w_s)$ be a reduced $1 \times (s+1)$ W-matrix and consider the W-weighted symmetric algebra $S^W(\mathcal{E})$: if $\mathcal{E}$ is locally free then $S^W(\mathcal{E})$ is locally free, too. The bundle $\mathbb{P}^W(\mathcal{E}) \to X'$, defined by setting

$$\mathbb{P}^W(\mathcal{E}) := \mathbf{Proj}(S^W(\mathcal{E})) = \mathbf{Proj}\left(S^W\left(\bigoplus_{k=0}^{s} \mathcal{O}_{X'}(E_k) \right)\right)$$

is called *the* (W-)*weighted projective toric bundle* (*WPTB*) *associated with* $\mathcal{E}$. Its fibers look like the s-dimensional weighted projective space $\mathbb{P}(w_0, \ldots, w_s)$ and it turns out to be a $\mathbb{Q}$-factorial complete toric variety whose fan is described as follows.

Let $\Sigma_W \subset N_{W,\mathbb{R}} \cong \mathbb{R}^s$ be a fan of $\mathbb{P}(W)$. Then its 1-skeleton $\Sigma_W(1)$ is composed by $s+1$ rays whose primitive generators are $s+1$ integer vectors $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_s \in \mathbb{Z}^s$ such that

$$\sum_{k=0}^{s} w_k \mathbf{e}_k = 0 \quad \text{and} \quad |\det(\mathbf{e}_0, \ldots, \widehat{\mathbf{e}}_i, \ldots, \mathbf{e}_s)| = w_i \text{ for } 0 \leq i \leq s,$$

see e.g. [21, Theorem 3]. The fan Σ_W is then composed by the cones

$$F_i := \langle \mathbf{e}_0, \mathbf{e}_1 \ldots, \widehat{\mathbf{e}}_i, \ldots, \mathbf{e}_s \rangle \subset \mathbb{R}^s, \quad 1 \leq i \leq s, \tag{13}$$

and all their faces. Consider now the fan defining X', given by $\Sigma' \subset N_{X',\mathbb{R}} \cong \mathbb{R}^{n'}$. Let $V' = (\mathbf{v}'_1, \ldots, \mathbf{v}'_{n'+r'})$ be an $n' \times (n'+r')$ fan matrix of X'. Let D'_j be the torus invariant Weil divisor associated with the ray $\langle \mathbf{v}'_j \rangle \in \Sigma(1)$. Then

$$E_k = \sum_{j=1}^{n'+r'} a_{kj} D'_j \quad \text{for all } k = 0, \ldots, s.$$

For a cone $\sigma \in \Sigma'$ define the *fibred* cone

$$\sigma_i := \left\langle \left\{ \begin{pmatrix} \mathbf{v}'_j \\ \mathbf{0}_{s,1} \end{pmatrix} + \sum_{k=0}^{s} a_{kj} \begin{pmatrix} \mathbf{0}_{n',1} \\ \mathbf{e}_k \end{pmatrix} \mid \langle \mathbf{v}'_j \rangle \in \sigma(1) \right\} \right\rangle + F_i \subset N_{X',\mathbb{R}} \times N_{W,\mathbb{R}} \cong \mathbb{R}^{n'+s}. \tag{14}$$

Proposition 2.16. *The set of fibred cones* (14) *and all their faces give rise to a fan* $\Sigma_{W,\mathcal{E}} \subset N_{X',\mathbb{R}} \times N_{W,\mathbb{R}}$ *whose toric variety is the* W*-weighted projective toric bundle* $\mathbb{P}^W(\mathcal{E})$.

The fibred cone (14) is the analogue of the cone (7.3.3) in [9] giving the fan of a projective toric bundle. The proof of the previous proposition is then the same as for [9, Proposition 7.3.3].

Let V be a fan matrix of $\mathbb{P}^W(\mathcal{E})$: setting $r = r' + 1$ and $n = n' + s$, by (14), V can be chosen to be the following $n \times (n+r)$ matrix

$$V = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\quad\quad\quad V' \quad\quad\quad}^{n'+r'}} & \overbrace{\mathbf{0}}^{s+1} \\ \sum_{k=0}^{s} a_{k,1} e_{1k} & \cdots & \sum_{k=0}^{s} a_{k,n'+r'} e_{1k} & \\ & \cdots & & \mathbf{e}_0 \cdots \mathbf{e}_s \\ \sum_{k=0}^{s} a_{k,1} e_{sk} & \cdots & \sum_{k=0}^{s} a_{k,n'+r'} e_{sk} & \end{array} \right) \tag{15}$$

By Gale duality, a weight matrix of $\mathbb{P}^W(\mathcal{E})$ is then given by the following $r \times (n+r)$ matrix

$$Q = \mathcal{G}(V) = \left(\begin{array}{cc} \overbrace{\begin{array}{c} Q' \\ 0 \cdots 0 \end{array}}^{n'+r'} & \overbrace{\begin{array}{c} Q'' \\ w_0 \cdots w_s \end{array}}^{s+1} \end{array} \right) \tag{16}$$

where $Q' = \mathcal{G}(V')$ and the $r' \times (s+1)$ matrix Q'' is defined by observing that

$$Q' V''^T + Q'' \begin{pmatrix} \mathbf{e}_0^T \\ \vdots \\ \mathbf{e}_s^T \end{pmatrix} = 0$$

where V'' is the $s \times (n'+r')$ matrix whose (i,j)-entry is given by $\sum_{k=0}^{s} a_{k,j} e_{ik}$. Therefore $Q'' = (b_{h,k+1})$ with

$$b_{h,k+1} = - \sum_{j=1}^{n'+r'} q'_{hj} a_{kj} \quad \text{for } 0 \le k \le s,$$

where $(q'_{h,j}) = Q'$. Recalling the morphism $d' : \mathcal{W}_T(X'(\Sigma')) \to \mathrm{Cl}(X'(\Sigma'))$ introduced in Proposition 1.11, this means that the $(k+1)$-st column of Q'' is given by

$$\mathbf{b}_{k+1} = -d'(E_k) \quad \text{for } 0 \le k \le s, \tag{17}$$

where $d'(E_k)$ is the class of E_k in $\mathrm{Cl}(X')$.

2.2.2 Toric covers. Recall that, given two lattices N and $\widetilde{N}$ with two fans $\Sigma \subset N_{\mathbb{R}}$ and $\widetilde{\Sigma} \subset \widetilde{N}_{\mathbb{R}}$, a $\mathbb{Z}$-linear map $\overline{f} : N \to \widetilde{N}$ is called *compatible* with the given fans if

$$\forall \sigma \in \Sigma \ \exists \widetilde{\sigma} \in \widetilde{\Sigma} \ : \ \overline{f}_{\mathbb{R}}(\sigma) \subseteq \widetilde{\sigma},$$

where $\overline{f}_{\mathbb{R}} : N_R \to \widetilde{N}_R$ is the natural $\mathbb{R}$-linear extension of $\overline{f}$; see [9, Definition 3.3.1].

For the following notion of *toric cover* we refer to [1, § 3].

Definition 2.17. A *toric cover* $f : X(\Sigma) \to \widetilde{X}(\widetilde{\Sigma})$ is a finite morphism of toric varieties inducing a $\mathbb{Z}$-linear map $\overline{f} : N \to \widetilde{N}$, compatible with Σ and $\widetilde{\Sigma}$, such that:

(1) $\overline{f}(N) \subseteq \widetilde{N}$ is a subgroup of finite index, so that $\overline{f}(N) \otimes \mathbb{R} = \widetilde{N} \otimes \mathbb{R}$,
(2) $\overline{f}_{\mathbb{R}}(\Sigma) = \widetilde{\Sigma}$.

Lemma 2.18 ([1] Lemma 3.3). *A toric cover $f : X(\Sigma) \to \widetilde{X}(\widetilde{\Sigma})$ has the following properties:*

(1) *f is an* abelian cover *with Galois group $G \cong \widetilde{N}/\overline{f}(N)$;*
(2) *f is ramified only along the torus invariant divisors D_ρ, with multiplicities $d_\rho \ge 1$ defined by the condition that the integral generator of $\overline{f}(N) \cap \langle \mathbf{v}_\rho \rangle$ is $d_\rho \mathbf{v}_\rho$, for every ray $\rho = \langle \mathbf{v}_\rho \rangle \in \widetilde{\Sigma}(1)$.*

2.2.3 Weighted Projective Toric weak Bundles (WPTwB). First note that Proposition 2.16 holds regardless of whether the divisors $E_k = \sum_{j=1}^{n'+r'} a_{kj} D'_j$, for $0 \le k \le s$, are truly Cartier divisors or instead, more generally, Weil divisors. Therefore the following natural question arises: which kind of geometric structures supports the toric variety associated with the simplicial complete fan given by Proposition 2.16 in the case of Weil non-Cartier divisors E_k's? The answer gives a nice account of both the previous subsections 2.2.1 and 2.2.2.

Recall that, given a Weil divisor D on a $\mathbb{Q}$-factorial variety, the *Cartier index* of D is the least positive integer $c(D) \in \mathbb{N}$ such that $c(D)D$ is a Cartier divisor.

Proposition 2.19. *Let V' be an $n' \times (n'+r')$ CF-matrix and $\Sigma' \in \mathcal{SF}(V')$. Consider the set Σ of fibred cones* (14) *and all their faces and assume that*

$$\forall\, 0 \le k \le s \ : \ E_k = \sum_{j=1}^{n'+r'} a_{kj} D'_j \in \mathcal{W}_T(X') \quad \textit{whose Cartier index is } l_k := c(E_k).$$

Then Σ is a simplicial and complete fan whose associated toric variety $X(\Sigma)$ is a toric cover of the WPTB $\mathbb{P}^{W'}(\mathcal{E})$, where W' is the reduced weight vector of $(l_0w_0,\dots,l_sw_s)$ and $\mathcal{E}=\bigoplus_{k=0}^s \mathcal{O}_{X'}(\eta_kE_k)$, with η_k defined by setting

$$\begin{aligned}
\lambda &:= \gcd(l_0w_0,\dots,l_sw_s)=\gcd(l_0,\dots,l_s)\quad (\text{since } \gcd(w_0,\dots,w_s)=1)\\
d_k &:= \gcd\Big(\frac{l_0w_0}{\lambda},\dots,\widehat{\frac{l_kw_k}{\lambda}},\dots,\frac{l_sw_s}{\lambda}\Big)=\gcd\Big(\frac{l_0}{\lambda},\dots,\widehat{\frac{l_k}{\lambda}},\dots,\frac{l_s}{\lambda}\Big)\quad (\text{since } W \text{ is reduced})\\
a_k &:= \operatorname{lcm}(d_0,\dots,\widehat{d}_k,\dots,d_s)=\frac{\prod_{i=0}^s d_i}{d_k}\quad (\text{by [21, Proposition 3(2)]})\\
a &:= \operatorname{lcm}(a_0,\dots,a_s)=\prod_{k=0}^s d_k\quad (\text{by [21, Proposition 3(5)]})\\
\eta_k &:= l_ka/a_k=l_kd_k.
\end{aligned}$$

In particular the toric cover $X(\Sigma)\to\mathbb{P}^{W'}(\mathcal{E})$ is an abelian covering admitting a Galois group G of order

$$|G|=\frac{1}{\lambda}\prod_{k=0}^s l_k$$

and ramified along the torus invariant divisors $D_{n'+r'+1+k}$, with multiplicity η_k, for $0\le k\le s$. In the following $X(\Sigma)$ is called a weighted projective toric weak bundle (WPTwB)*: it is a PWS.*

Proof. As for the proof of Proposition 2.16, the fact that Σ is a simplicial and complete fan follows by the same argument proving [9, Proposition 7.3.3].

Given Cartier indexes $l_k=c(E_k)\ge 1$, consider the diagonal matrix

$$\Lambda' := \begin{pmatrix} I_{n'+r'} & \mathbf{0}_{n'+r',s+1}\\ \mathbf{0}_{s+1,n'+r'} & \operatorname{diag}(l_0,\dots,l_s)\end{pmatrix}\in \mathrm{GL}_{n+r}(\mathbb{Q})\cap \mathbf{M}_{n+r}(\mathbb{Z}).$$

Then, recalling (16) and (17), one gets

$$Q\cdot\Lambda' = \Bigg(\overbrace{\begin{matrix} Q'\\ 0\cdots 0\end{matrix}}^{n'+r'}\ \overbrace{\begin{matrix} -d(l_0E_0) & \cdots & -d(l_sE_s)\\ l_0w_0 & \cdots & l_sw_s\end{matrix}}^{s+1}\Bigg)$$

If the weight vector $(l_0w_0,\dots,l_sw_s)$ is reduced, then set $W'=(l_0w_0,\dots,l_sw_s)$ and $Q\cdot\Lambda'$ turns out to be a weight matrix of $\mathbb{P}^{W'}(\bigoplus_{k=0}^s\mathcal{O}_{X'}(l_kE_k))$.

If $(l_0w_0,\dots,l_sw_s)$ is not reduced, then define λ, d_k, a_k and a as in the statement and consider the matrices

$$\begin{aligned}
\Delta &:= \operatorname{diag}\Big(1,\dots,1,\frac{1}{\lambda a}\Big)\in \mathrm{GL}_r(\mathbb{Q})\\
\Lambda'' &:= \begin{pmatrix} I_{n'+r'} & \mathbf{0}_{n'+r',s+1}\\ \mathbf{0}_{s+1,n'+r'} & \operatorname{diag}(\frac{a}{a_0},\dots,\frac{a}{a_s})\end{pmatrix}\in \mathrm{GL}_{n+r}(\mathbb{Q})\cap \mathbf{M}_{n+r}(\mathbb{Z})\\
\Lambda &:= \Lambda'\cdot\Lambda'' = \begin{pmatrix} I_{n'+r'} & \mathbf{0}_{n'+r',s+1}\\ \mathbf{0}_{s+1,n'+r'} & \operatorname{diag}(\frac{l_0a}{a_0},\dots,\frac{l_sa}{a_s})\end{pmatrix}\\
\widetilde{Q} &:= \Delta\cdot Q\cdot\Lambda = \begin{pmatrix} Q' & -d(\frac{l_0a}{a_0}E_0) & \cdots & -d(\frac{l_sa}{a_s}E_s)\\ 0\cdots 0 & \frac{l_0w_0}{\lambda a_0} & \cdots & \frac{l_sw_s}{\lambda a_s}\end{pmatrix} = \begin{pmatrix} Q' & -d(\eta_0E_0) & \cdots & -d(\eta_sE_s)\\ 0\cdots 0 & w'_0 & \cdots & w'_s\end{pmatrix}
\end{aligned}$$

where $W'=(w'_0,\dots,w'_s)$ is the reduced weight vector of W and $\eta_k=l_ka/a_k$. Then $\widetilde{Q}$ turns out to be a weight matrix of $\mathbb{P}^{W'}(\bigoplus_{k=0}^s\mathcal{O}_{X'}(\eta_kE_k))$.

Recalling (15), the fan matrix V is a CF-matrix since V' is a CF-matrix. Then $X(\Sigma)$ is a PWS and $\mathrm{Cl}(X)$ is a free $\mathbb{Z}$-module. Then the dualized divisors' exact sequence (7) is exact on the right, too, hence giving the following short exact sequence of free abelian groups

$$0 \longrightarrow \operatorname{Hom}(\mathrm{Cl}(X),\mathbb{Z}) \xrightarrow{d^\vee} \operatorname{Hom}(\mathcal{W}_T(X),\mathbb{Z}) \xrightarrow{div^\vee} N \longrightarrow 0\,. \tag{18}$$

Fixing once for all a basis of $M \cong \mathbb{Z}^n$, the basis $\{D_j\}_{j=0}^{n+r}$ of $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$, a basis of $F^r = \mathrm{Cl}(X) \cong \mathbb{Z}^r$ and their dual bases, then Q^T and V turn out to be representative matrices of morphisms $d^\vee$ and $div^\vee$, respectively. Then we get the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{Cl}(X),\mathbb{Z}) \cong \mathbb{Z}^r \arrow[r, "(\Lambda^{-1})^T"] \arrow[d, "Q^T", "d^\vee"'] & \mathbb{Z}^r \arrow[r] \arrow[d, "\widetilde{Q}^T"] & \mathbb{Z}/(\lambda a)\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{W}_T(X),\mathbb{Z}) \cong \mathbb{Z}^{n+r} \arrow[r, "\Lambda^T"] \arrow[d, "V", "div^\vee"'] & \mathbb{Z}^{n+r} \arrow[r] \arrow[d, "\widetilde{V}"] & (\bigoplus_{k=0}^s \mathbb{Z}/\eta_k\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \cong \mathbb{Z}^n \arrow[r, "\Phi^T", "\overline{f}"'] \arrow[d] & N_{X'} \times N_{W'} \cong \mathbb{Z}^n \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{19}$$

where the matrix Φ is obtained as follows:

- V is a CF-matrix if and only if $H := \mathrm{HNF}(V^T) = \begin{pmatrix} I_n \\ \mathbf{0}_{r,n} \end{pmatrix}$, see [23, Theorem 2.1(4)],
- let $U \in \mathrm{GL}_{n+r}(\mathbb{Z})$ such that $U \cdot V^T = H$,
- then the upper n rows of U give ${}^nU \cdot V^T = I_n$ (recall notation in list 1.1)

Therefore

$$V^T \cdot \Phi = \Lambda \cdot \widetilde{V}^T \quad \Longrightarrow \quad \Phi = {}^nU \cdot \Lambda \cdot \widetilde{V}^T.$$

From diagram (19), the $\mathbb{Z}$-linear morphism $\overline{f} : N \to N_{X'} \times N_{W'}$ represented by Φ is clearly injective, giving rise to the toric cover we were looking for, whose Galois group is given by G in the same diagram. The exactness of the vertical sequence on the right implies that

$$|G| = \frac{1}{\lambda a} \prod_{k=0}^{s} \eta_k = \frac{1}{\lambda a} \prod_{k=0}^{s} l_k d_k = \frac{1}{\lambda} \prod_{k=0}^{s} l_k,$$

while the ramification is given by the matrix $\Lambda^T = \Lambda$. □

Remark 2.20. The hypothesis that V' is a CF-matrix is essential in proving Proposition 2.19. In fact, recalling (15) if V' is an F non-CF-matrix, then V is an F non-CF-matrix, too. Then the dual exact sequence (7) is not exact on the right, meaning that the morphism $\overline{f}$ in diagram (19) may not exist.

On the other hand the set Σ of fibred cones (14) and all their faces still turns out to be a simplicial complete fan, due to the same argument proving [9, Proposition 7.3.3]. It remains then open to give a geometric interpretation of this case, for which the reader is referred to § 3 and in particular to Theorem 3.4 and Remark 3.6.

2.3 Maximally bordering chambers and WPTB. The present subsection generalizes Batyrev's results given in [2, § 4], by dropping the smoothness hypothesis. First note the following useful fact:

Lemma 2.21. *Let V and $\widetilde{V}$ be $n \times (n+r)$ reduced F-matrices such that $f : X(\Sigma) \to \widetilde{X}(\widetilde{\Sigma})$ is a toric cover for some $\Sigma \in \mathcal{SF}(V)$ and $\widetilde{\Sigma} \in \mathcal{SF}(\widetilde{V})$. Then $\gamma_\Sigma \in \Gamma(V)$ is a maxbord chamber if and only if $\widetilde{\gamma}_{\widetilde{\Sigma}} \in \Gamma(\widetilde{V})$ is a maxbord chamber.*

Proof. Recalling the Definition 2.17 of a toric cover, the induced $\mathbb{Z}$-linear morphism $\overline{f} : N \to \widetilde{N}$, which is compatible with the fans Σ and $\widetilde{\Sigma}$, gives actually an equality of fans $\overline{f}_{\mathbb{R}}(\Sigma) = \widetilde{\Sigma}$. Then for every cone $\sigma \in \Sigma$ define $\widetilde{\sigma} := \overline{f}_{\mathbb{R}}(\sigma) \in \widetilde{\Sigma}$. By Theorem 2.11, there exists a bordering primitive collection $\mathcal{P} = \{V_P\}$ of Σ, for some $P \in \mathfrak{P}$, whose support hyperplane H_P is the maximally bordering hyperplane of γ. Then, by Proposition 2.8,

$$\forall \langle V_I \rangle \in \Sigma(n)\ \exists!\, i \in P\ :\ \langle Q^I \rangle = \langle \mathbf{q}_i \rangle + \langle Q^I \rangle \cap H_P \Longleftrightarrow \langle V_I \rangle = \langle V_{P\setminus\{i\}} \rangle + \langle V_{I\setminus P} \rangle.$$

On the other hand $\widetilde{\mathcal{P}} := \{\overline{f}_{\mathbb{R}}(\mathbf{v}_i) \mid i \in P\} = \{\widetilde{V}_{\widetilde{P}}\}$ turns out to be a primitive collection for $\widetilde{\Sigma} = \overline{f}_{\mathbb{R}}(\Sigma)$. Up to a permutation of columns of $\widetilde{V}$ we can assume $\widetilde{P} = P$. Then

$$\begin{aligned}\forall \langle \widetilde{V}_I\rangle \in \widetilde{\Sigma}(n)\ \exists!\, i \in P\ :\ \langle \widetilde{V}_I\rangle = \overline{f}_{\mathbb{R}}(\langle V_I\rangle) = \overline{f}_{\mathbb{R}}(\langle V_{P\backslash\{i\}}\rangle) + \overline{f}_{\mathbb{R}}(\langle V_{I\backslash P}\rangle) = \langle \widetilde{V}_{P\backslash\{i\}}\rangle + \langle \widetilde{V}_{P\backslash I}\rangle \\ \iff \quad \langle \widetilde{Q}^I\rangle = \langle \widetilde{\mathbf{q}}_i\rangle + (\langle \widetilde{Q}^I\rangle \cap \widetilde{H}_P)\end{aligned}$$

which is enough, by Proposition 2.8, to show that $\widetilde{\gamma}_{\widetilde{\Sigma}}$ is maxbord with respect to the support $\widetilde{H}_P$ of $\widetilde{\mathcal{P}}$.

The converse can be proved in the same way by observing that $\Sigma = \overline{f}_{\mathbb{R}}^{-1}(\widetilde{\Sigma})$. □

We are now in a position to state and prove the following generalization of [2, Proposition 4.1].

Theorem 2.22. *Given a reduced $n\times(n+r)$ CF-matrix V with $r \geq 2$, a chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is maximally bordering if and only if the associated PWS $X(\Sigma_\gamma)$ is a toric cover of a weighted projective toric bundle $\mathbb{P}^W(\mathcal{E})$.*

Proof. Recalling Definition 2.5 and Lemma 2.14 we can assume that γ is a maxbord chamber with respect to H_r, which gives $\dim(\gamma \cap H_r) = r-1$. Since γ is maxbord, it is intbord and Theorem 2.11 implies that, after suitable transformations, the reduced W-matrix $Q = \mathcal{G}(V)$ can be set in REF with the bottom r-th row giving a primitive relation for Σ_γ, $\mathcal{P} = \{\mathbf{v}_{n+r-s}, \dots, \mathbf{v}_{n+r}\}$, on the last $s+1$ columns of V (here we are exchanging the roles of s and $n+r-s$ with respect to the proof of Theorem 2.11). Then Q looks like (16) where Q' is an $(r-1)\times(n+r-s-1)$ matrix in REF.

First note that Q' can be thought of as a W-matrix of an $n' = (n-s)$-dimensional variety, with the exception of condition b in Definition 1.3. In fact, the REF form of Q and the fact that Q is a W-matrix imply immediately conditions a, c and d of Definition 1.3 for Q'.

Concerning condition e, we observe that $\mathcal{L}_r(Q')$ cannot contain any vector of the form $(0, \dots, 0, q, 0, \dots, 0)$. Otherwise, if the non-trivial entry q is in the i-th position then the i-th column $\mathbf{q}'_i$ of Q' cannot be in $\mathcal{L}_c(Q'^{\{i\}})$. Therefore $\dim\langle Q'^{\{i\}}\rangle \leq r-2$. On the other hand, by the REF of Q and (1), one gets

$$\gamma \cap H_r \subseteq \mathrm{Mov}(V) \cap H_r \subseteq \langle Q^{\{i\}}\rangle \cap H_r = \langle Q'^{\{i\}}\rangle \tag{20}$$

where the last equality on the right comes from the fact that γ is maxbord with respect to H_r, meaning that H_r cuts a facet of $\mathrm{Mov}(V)$, hence a facet of $\langle Q^{\{i\}}\rangle$. Clearly (20) contradicts the maxbord hypothesis $\dim(\gamma \cap H_r) = r-1$.

The same argument applies to guarantee condition f of Definition 1.3 for Q'. In fact $\mathcal{L}_r(Q')$ cannot contain any vector of the form $(0, \dots, 0, a, 0, \dots, 0, b, 0, \dots, 0)$ with $ab < 0$. Otherwise, if the non-trivial entries a, b are in the i-th and j-th positions, respectively, then the i-th and the j-th columns $\mathbf{q}'_i, \mathbf{q}'_j$ of Q' cannot be in $\mathcal{L}_c(Q'^{\{i,j\}})$. Therefore $\dim\langle Q'^{\{i,j\}}\rangle \leq r-2$. Moreover one also gets that

$$\forall \mu, \lambda \quad \mu\lambda > 0 \quad \Longrightarrow \quad \mu\mathbf{q}_i - \lambda\mathbf{q}_j \notin \mathcal{L}_c(Q'^{\{i,j\}}) \tag{21}$$

because $\mu a - \lambda b \neq 0$. On the other hand, by the REF of Q and (1), one gets

$$\gamma \cap H_r \subseteq \mathrm{Mov}(V) \cap H_r \subseteq \langle Q^{\{i\}}\rangle \cap \langle Q^{\{j\}}\rangle \cap H_r = \langle Q'^{\{i\}}\rangle \cap \langle Q'^{\{j\}}\rangle \tag{22}$$

where the last equality on the right comes from the fact that H_r cuts a facet of both $\langle Q^{\{i\}}\rangle$ and $\langle Q^{\{j\}}\rangle$. Note that if one proves that

$$\langle Q'^{\{i\}}\rangle \cap \langle Q'^{\{j\}}\rangle = \langle Q'^{\{i,j\}}\rangle \tag{23}$$

then (22) turns out to contradict the maxbord hypothesis $\dim(\gamma \cap H_r) = r-1$. Since clearly $\langle Q'^{\{i\}}\rangle \cap \langle Q'^{\{j\}}\rangle \supseteq \langle Q'^{\{i,j\}}\rangle$, to show (23) we need to prove that if $\mathbf{x} \in \langle Q'^{\{i\}}\rangle \cap \langle Q'^{\{j\}}\rangle$ then $\mathbf{x} \in \langle Q'^{\{i,j\}}\rangle$. For this purpose consider the linear combinations with nonnegative coefficients

$$\mathbf{x} = \sum_{k\neq i,j} \lambda_k\mathbf{q}_k + \lambda_j\mathbf{q}_j = \sum_{k\neq i,j} \mu_k\mathbf{q}_k + \mu_i\mathbf{q}_i \in \langle Q'^{\{i\}}\rangle \cap \langle Q'^{\{j\}}\rangle.$$

This gives $\mu_i\mathbf{q}_i - \lambda_j\mathbf{q}_j = \sum_{k\neq i,j}(\lambda_k - \mu_k)\mathbf{q}_k$, contradicting (21) unless $\mu_i = \lambda_j = 0$, which is $\mathbf{x} \in \langle Q'^{\{i,j\}}\rangle$.

We have now to consider three possible cases: (a) Q' is a reduced W-matrix, (b) Q' is a non-reduced W-matrix, (c) Q' is not a W-matrix in the sense that $\mathcal{L}_r(Q')$ has cotorsion in $\mathbb{Z}^{n'+r'}$, with $n' = n - s$ and $r' = r - 1$.

(a) Assume that Q' is a reduced W-matrix. Since γ is maxbord, $\gamma' := \gamma \cap H_r$ is $(r-1)$-dimensional. We show that γ' is actually a chamber contained in $\mathrm{Mov}(V')$, with $V' = \mathcal{G}(Q')$. For this purpose note that (1) and the fact that H_r cuts out a facet of every $\langle Q^{\{i\}}\rangle$ give

$$\mathrm{Mov}(V') = \bigcap_{i=1}^{n+r-s-1} \langle Q'^{\{i\}}\rangle = H_r \cap \bigcap_{i=1}^{n+r} \langle Q^{\{i\}}\rangle = H_r \cap \mathrm{Mov}(V).$$

Hence $\gamma' = H_r \cap \gamma \subset H_r \cap \mathrm{Mov}(V) = \mathrm{Mov}(V')$. Finally observe that by Proposition 2.8 every simplicial cone $\langle Q_I\rangle$ in the bunch of cones $\mathcal{B}(\gamma)$ has to contain the $(r-1)$-dimensional chamber $\gamma' = \gamma \cap H_r$, hence cutting out a simplicial cone $\langle Q_I\rangle \cap H_r =: \langle Q'_{I'}\rangle \in \mathcal{B}(\gamma')$ and admitting a unique ray generated by a column $\mathbf{q}_i$ of Q not belonging to H_r i.e.

$$\mathbf{q}_i \in \mathcal{P}^*, \quad I = I' \cup \{i\}, \quad \langle Q_I\rangle = \langle Q'_{I'}\rangle + \langle \mathbf{q}_i\rangle,$$

with $i \geq n + r - s$. Let us now consider Gale dualities with respect to W-matrices Q, Q' and W, giving the Gale dual cones $\mathcal{G}(\langle Q_I\rangle) = \langle V^I\rangle$, $\mathcal{G}(\langle Q'_{I'}\rangle) = \langle (V')^{I'}\rangle$ and $\mathcal{G}(\langle \mathbf{q}_i\rangle) = F_i$, respectively, where the latter is precisely the cone F_i defined in (13). Then $\langle V^I\rangle = \langle (V')^{I'}\rangle + F_i$. Note that, on the one hand, the set of cones $\langle (V')^{I'}\rangle$ and all their faces define an $(n-s)$-dimensional fan $\Sigma'_{\gamma'}$ and, on the other hand, the cones F_i, jointly with all their faces, give the fan Σ_W of $\mathbb{P}(W)$. This suffices to show that Σ_γ is *split by* $\Sigma'_{\gamma'}$ *and* Σ_W, in the sense of [9, Definition 3.3.18]. Therefore [9, Theorem 3.3.19] gives a locally trivial fibre bundle $X(\Sigma_\gamma) \twoheadrightarrow X'(\Sigma'_{\gamma'})$ whose fibers are all isomorphic to $\mathbb{P}(W)$.

It remains to prove that such a fiber bundle is actually a WPTwB, as defined in 2.2.3, hence a toric cover of a WPTB associated with some locally free sheaf $\mathcal{E}$. For this purpose add suitable negative multiples of the bottom row of Q to the previous ones until one gets no positive entries in the (i,j)-positions with $1 \leq i \leq r-1$ and $n + r - s \leq j \leq n + r$. These entries give the matrix Q'' in (16) whose columns give, up to a sign, the linear equivalence classes of some Weil divisors $E_0, \dots, E_s$, as in (17). Consequently the Gale dual $\langle V^I\rangle$ of every cone $\langle Q_I\rangle \in \mathcal{B}(\gamma)$ turns out to be a fibred cone (14). Recalling [22, Proposition 3.12(1)], $V' = \mathcal{G}(Q')$ is a CF-matrix. This suffices to show that $X(\Sigma_\gamma)$ is a WPTwB, by Proposition 2.19. Then $X(\Sigma_\gamma)$ is a toric cover of $\mathbb{P}^{W'}(\bigoplus_{k=0}^{s} \mathcal{O}_{X'(\Sigma'_{\gamma'})}(\eta_k E_k))$, where W' and η_k are defined as in Proposition 2.19. Note that, by (15), V' is a CF-matrix if and only if V is a CF-matrix: then this Case (a) can occur only if V is a CF-matrix, as assumed in the statement.

(b) Assume now that Q' is a non-reduced W-matrix (hence $\mathcal{L}_r(Q')$ has not cotorsion in $\mathbb{Z}^{n'+r'}$). Then $V' = \mathcal{G}(Q')$ admits some non-primitive column. Without loss of generality, up to a permutation of columns and an iteration of the following argument, assume that the unique non-primitive column of V' is the first one: namely $\mathbf{v}_1 = d\mathbf{w}_1$, with $\mathbf{w}_1$ primitive. Consider the reduced weight matrix $Q'^{\mathrm{red}} = \mathcal{G}(V'^{\mathrm{red}})$, see the list of notation 1.1. The construction described in [22, Theorem 3.15(3)] then gives

$$Q'^{\mathrm{red}} = \mathrm{diag}(1, \dots, 1, 1/d)\alpha_1 Q' \,\mathrm{diag}(d, 1, \dots, 1)$$

for a suitable $\alpha_1 \in \mathrm{GL}_{r-1}(\mathbb{Z})$. Define

$$\begin{aligned} A &= \begin{pmatrix} \mathrm{diag}(1, \dots, 1, 1/d)\alpha_1 & \mathbf{0}_{r-1,1} \\ \mathbf{0}_{1,r-1} & 1/d \end{pmatrix} \in \mathrm{GL}_r(\mathbb{Q}) \\ B &= \begin{pmatrix} \mathrm{diag}(d, 1, \dots, 1) & \mathbf{0}_{n+r-s-1,s+1} \\ \mathbf{0}_{s+1,n+r-s-1} & dI_{s+1} \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z}) \\ \widetilde{Q} &= AQB \in \mathbf{M}(r, n+r; \mathbb{Z}). \end{aligned} \tag{24}$$

Note that the bottom r-th row of $\widetilde{Q}$ coincides with the bottom r-th row of Q. Then $\widetilde{Q}$ is reduced for the REF of Q and the fact that Q'^{red} is reduced. Define $\widetilde{V} = (\widetilde{\mathbf{v}}_1, \dots, \widetilde{\mathbf{v}}_{n+r}) := \mathcal{G}(\widetilde{Q})$. Consider the dual divisors' sequence (7) for $X(\Sigma_\gamma)$ and note that it turns out to be exact on the right, since V is a CF-matrix, hence giving the short exact sequence (18). Fixing the bases of the $\mathbb{Z}$-modules appearing in (18), the matrices defined in (24) define $\mathbb{Z}$-linear morphisms giving the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{Cl}(X)) \cong \mathbb{Z}^r \arrow[r, "(A^{-1})^T"] \arrow[d, "d^\vee"', "Q^T"] & \mathbb{Z}^r \arrow[r] \arrow[d, "\widetilde{Q}^T"] & (\mathbb{Z}/d\mathbb{Z})^{\oplus 2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{W}_T(X)) \cong \mathbb{Z}^{n+r} \arrow[r, "B^T"] \arrow[d, "div^\vee"', "V"] & \mathbb{Z}^{n+r} \arrow[r] \arrow[d, "\widetilde{V}"] & (\mathbb{Z}/d\mathbb{Z})^{\oplus s+2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \cong \mathbb{Z}^n \arrow[r, "C^T"] \arrow[d] & \mathbb{Z}^n \arrow[r] \arrow[d] & (\mathbb{Z}/d\mathbb{Z})^{\oplus s} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{25}$$

where C^T is the representative matrix of an injective $\mathbb{Z}$-linear map $\overline{f} : \mathbb{Z}^n \to \mathbb{Z}^n$ easily defined by diagram chasing. Then

$$C^T V = \widetilde{V} B^T = (d\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_{n+r-s-1}, d\tilde{\mathbf{v}}_{n+r-s}, \dots, d\tilde{\mathbf{v}}_{n+r})$$

and $\overline{f}(N)$ is a subgroup of finite index d^s of $\mathbb{Z}^n = \widetilde{N}$, as deduced by the vertical sequence on the right. Moreover, by Lemma 2.21, $\widetilde{\Sigma} := \overline{f}_{\mathbb{R}}(\Sigma_\gamma)$ is a fan in $\mathbb{P}\mathcal{S}\mathcal{F}(\widetilde{V})$ defining a PWS $\widetilde{X}(\widetilde{\Sigma})$ whose weight matrix is $\widetilde{Q}$ and whose chamber $\widetilde{\gamma}_{\widetilde{\Sigma}} \in \Gamma(\widetilde{V})$ is maxbord. Then, by the previous Part (a), $\widetilde{X}$ is a WPTwB, which is a toric cover of a WPTB. Moreover $\overline{f}$ induces a toric cover $f : X \to \widetilde{X}$, hence X is a toric cover of a WPTB.

We note that in the present situation one can say something more than Lemma 2.21: in fact, the matrix A^{-1} represents an injective $\mathbb{Z}$-linear map $g : \mathrm{Cl}(\widetilde{X}) \to \mathrm{Cl}(X)$ which is compatible with the secondary fans $\Gamma(\widetilde{V})$ and $\Gamma(V)$ and gives $g_{\mathbb{R}}(\Gamma(\widetilde{V})) = \Gamma(V)$.

(c) Finally we now assume that $\mathcal{L}_r(Q')$ has cotorsion in $\mathbb{Z}^{n'+r'}$. Without loss of generality, up to a permutation of rows and an iteration of the following argument, assume that $\alpha Q'$, for some $\alpha \in \mathrm{GL}_{r'}(\mathbb{Z})$, admits a unique row giving cotorsion, namely the bottom r'-th row of Q'. Let $d > 1$ be the greatest common divisor of all entries in that row, i.e. $d = \gcd(q_{r',1}, \dots, q_{r',n'+r'})$. Recall the matrix A given in (24) and define

$$\begin{aligned}
A' &= A \begin{pmatrix} \alpha & \mathbf{0}_{r',1} \\ \mathbf{0}_{1,r'} & 1 \end{pmatrix} \in \mathrm{GL}_r(\mathbb{Q}) \\
B &= \begin{pmatrix} I_{n'+r'} & \mathbf{0}_{n'+r',s+1} \\ \mathbf{0}_{s+1,n'+r'} & dI_{s+1} \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z}) \\
\widetilde{Q} &= A' Q B \in \mathbf{M}(r, n+r; \mathbb{Z}).
\end{aligned}$$

Clearly the bottom r-th row of $\widetilde{Q}$ coincides with the bottom r-th row of Q. Moreover $\widetilde{Q}$ turns out to be a W-matrix for the REF of Q and the fact that $\widetilde{Q}' = \mathrm{diag}(1, \dots, 1, 1/d)\alpha_1 \alpha Q'$ is now a W-matrix. From this we can go on as in Part (b), showing that X is a toric cover of a suitable WPTB.

For the converse, let us assume that $X(\Sigma_\gamma)$ is a toric cover of a WPTB $\mathbb{P}^W(\mathcal{E})$. This means that there exists a $\mathbb{Z}$-linear morphism $\overline{f} : N \to \widetilde{N}$ such that $\widetilde{\Sigma} = \overline{f}_{\mathbb{R}}(\Sigma_\gamma)$ is the fan of $\widetilde{X} = \mathbb{P}^W(\mathcal{E})$. Therefore $\widetilde{\Sigma}$ is composed of fibred cones (14) implying that it is split by an n'-dimensional fan $\widetilde{\Sigma}'$ and an s-dimensional fan $\widetilde{\Sigma}_W$. The equality of fans $\widetilde{\Sigma} = \overline{f}_{\mathbb{R}}(\Sigma_\gamma)$ then imposes an analogous splitting for the fan Σ_γ. Gale duality then gives that every cone in the bunch $\mathcal{B}(\gamma)$ is the sum of a 1-dimensional cone and an $(r-1)$-dimensional cone belonging to a fixed facet of the Gale dual cone $\mathcal{Q}$. Proposition 2.8 then enables us to conclude that γ is a maxbord chamber. □

The geometric picture described by the previous Theorem 2.22 dramatically simplifies in the case of smooth projective toric varieties: in this context, the following result is equivalent to [2, Proposition 4.1].

Corollary 2.23. *Given a reduced $n\times(n+r)$ CF-matrix V with $r \geq 2$, a chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is maximally bordering and non-singular if and only if the associated PWS $X(\Sigma_\gamma)$ is a projective toric bundle $\mathbb{P}(\mathcal{E}) \to X'$ over a smooth PWS $X'(\Sigma')$.*

Proof. Since maxbord implies intbord, Theorem 2.11 and Corollary 2.12 give a numerically effective primitive collection for Σ_γ whose primitive relation has all the non-zero coefficient equal to 1. By Lemma 2.14, such a primitive relation can be considered as the bottom row of the REF positive weight matrix Q. Hence Theorem 2.22 gives that X is a toric cover of a projective toric bundle $\mathbb{P}(\mathcal{E})$, since $W = (1, \dots, 1)$. In particular the covering map $f : X \to \mathbb{P}(\mathcal{E})$ is the identity if and only if the $r' \times (n' + r')$ matrix Q', obtained as in (16), is a reduced W-matrix. The fact that γ is a non-singular chamber implies that Q' is necessarily a reduced W-matrix: in other words Cases (b) and (c) in the proof of Theorem 2.22 cannot occur. In fact:

- for (c) note that Proposition 2.8 gives that every cone $\langle Q_I\rangle \in \mathcal{B}(\gamma)$ admits a unique generator $\mathbf{q}_i$, with $n' + r' + 1 \leq i \leq n + r$, such that $\langle Q_I\rangle = \langle \mathbf{q}_i\rangle + \langle Q'_{I\setminus\{i\}}\rangle$; by the REF of Q, $|\det(Q_I)| = q_{r,i}|\det(Q'_{I\setminus\{i\}})|$; since $\mathcal{L}_r(Q')$ has cotorsion in $\mathbb{Z}^{n'+r'}$, not every r-minor of Q' can be unimodular, giving $|\det(Q_I)| > 1$, against the non-singularity of γ;
- for (b) note that if Q' is a non-reduced W-matrix, then there exists a column index h such that $1 \leq h \leq n' + r'$ and $\mathcal{L}_r(Q'^{\{h\}})$ has cotorsion in $\mathbb{Z}^{n'+r'-1}$; in the bunch $\mathcal{B}(\gamma)$ there certainly exists a cone $\sigma^{\{h\}}$ not admitting the column $\mathbf{q}_h$ as a generator; then by Proposition 2.8, $\sigma^{\{h\}}$ admits a unique generator $\mathbf{q}_i$ with $n' + r' + 1 \leq i \leq n + r$; as above, the REF of Q then gives $|\det(\sigma^{\{h\}})| > 1$, against the non-singularity of γ.

Then X is a PTB $\mathbb{P}(\mathcal{E}) \to X'$. The smoothness of X' follows by the smoothness of X.

The converse is obvious, since a PTB $X(\Sigma_\gamma) = (\mathbb{P}(\mathcal{E}) \to X')$ over a smooth PWS X' is clearly smooth, giving the non-singularity of γ. Moreover γ is maxbord by Theorem 2.22. □

The previous results allows us to give the following characterization of a PWS which is a toric flip (in the sense of § 1.3) of a toric cover of a PWS, by means of a particular condition on the weight matrix: see the following Example 2.42 for a PWS not satisfying such a condition and then not realizing this kind of a birational equivalence.

Theorem 2.24. *Let V be a CF-matrix and consider $X(\Sigma)$, with $\Sigma \in \mathbb{P}\mathcal{SF}(V)$. Then X is a toric flip of a toric cover $\widetilde{X} \twoheadrightarrow \mathbb{P}^W(\mathcal{E})$ of a WPTB if and only if* Mov(V) *is maxbord with respect to an hyperplane $H \subseteq F^r_{\mathbb{R}}$, i.e. up to an application of Lemma* 2.14 *sending H to $H_r = \{x_r = 0\}$ there exists a positive, REF, W-matrix $Q = \mathcal{G}(V)$ looking as in* (16) *and such that*

1. *either the left-upper submatrix Q' is a reduced W-matrix: in this case X is a toric flip of a WPTwB;*
2. *or the left-upper submatrix Q' is either a non-reduced W-matrix or satisfies all the conditions of Definition* 1.3 *but condition b: in this case X is a toric flip of a toric cover of a WPTwB.*

Moreover, if X is smooth Case (2) *cannot occur and X turns out to be a toric flip of a PTB if and only if the left-upper submatrix Q' is a reduced W-matrix.*

Proof. Let $X(\Sigma)$ be a toric flip of a toric cover $\widetilde{X}(\widetilde{\Sigma})$ of a WPTB $\mathbb{P}^W(\mathcal{E})$. This means that we can assume that $\widetilde{\Sigma} \in \mathbb{P}\mathcal{SF}(V)$ and that $\widetilde{\gamma} := \gamma_{\widetilde{\Sigma}}$ is a maxbord chamber, with respect to an hyperplane H, of $\Gamma(V)$. This is enough to show that Mov(V) is maxbord with respect to H. By Lemma 2.14 we can assume that $Q = \mathcal{G}(V)$ is a positive, REF, W-matrix looking as in (16) and that $H = H_r$ is the supporting hyperplane of a bordering primitive collection for $\widetilde{\Sigma}$. Proceeding as in the proof of Theorem 2.22, the upper left submatrix Q' turns out to satisfy Conditions (1) and (2) of the statement.

Conversely, assume that Mov(V) is maxbord with respect to an hyperplane H. This means that there exists a maxbord chamber $\widetilde{\gamma} \subseteq$ Mov(V) with respect to the hyperplane H. Proceeding as in the proof of Theorem 2.22 we can assume that $H = H_r$ and that $Q = \mathcal{G}(V)$ is a positive, REF, W-matrix looking as in (16). In particular the upper left submatrix Q' turns out to satisfy Conditions (1) and (2) in the statement. Setting $\widetilde{\Sigma} := \Sigma_{\widetilde{\gamma}} \in \mathbb{P}\mathcal{SF}(V)$, Theorem 2.22 ensures that $\widetilde{X}(\widetilde{\Sigma})$ is either a WPTwB, when Q' satisfies Condition (1), or a toric cover of a WPTwB, when Q' satisfies Condition (2). Clearly X is a toric flip of $\widetilde{X}$.

The last part of the statement, regarding the smooth case, follows by Corollary 2.23. □

2.4 The geometric meaning of a maximally bordering chamber. Recall Proposition 1.11: a maxbord chamber γ with respect to a hyperplane H gives a fan $\Sigma = \Sigma_\gamma$ such that the hyperplane H cuts out a common facet of $\mathrm{Nef}(X(\Sigma))$ and $\overline{\mathrm{Eff}}(X(\Sigma))$: dually we are fixing an extremal ray of the Mori cone $\overline{\mathrm{NE}}(X_\Sigma)$. By [9, Proposition 15.4.1, Lemma 15.4.2(b,c) and Proposition 15.4.5(a)], contracting such an extremal ray gives rise to a fibering morphism of $\mathbb{Q}$-factorial complete toric varieties $\phi : X(\Sigma) \to X_0(\Sigma_0)$ whose fibers are connected and isomorphic to a finite abelian quotient of a WPS (also called a *fake WPS*) whose dimension is given by s, where $s + 1$ is the cardinality $|\mathcal{P}_H|$ of the primitive collection supported by H.

On the other hand if X is a PWS, then Theorem 2.22 exhibits X as a toric cover of a WPTB $\mathbb{P}^W(\mathcal{E})$ so giving

$$X \xrightarrow[\text{toric cover}]{f} \mathbb{P}^W(\mathcal{E}) \xrightarrow[\text{WPTB}]{\varphi} X' .$$

Putting all together this means that the fibering morphism ϕ gives the morphism with connected fibers of the Stein factorization of $\varphi \circ f$, which is

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & \mathbb{P}^W(\mathcal{E}) \arrow[d, "\varphi"] \\ X_0 \arrow[r, "f_0", "\text{finite}"'] & X' \end{tikzcd} \tag{26}$$

Let us underline that, by Theorem 2.22, the right hand side of diagram (26) allows one to completely determine (starting from a fan CF-matrix V and, by Gale duality, a REF positive W-matrix $Q = \mathcal{G}(V)$) the toric cover f, the WPS giving the fibers of $\mathbb{P}^W(\mathcal{E})$ and the basis X', in terms of a collection of matrices giving diagram (25).

Moreover Corollary 2.23 shows that when X is smooth both the finite morphisms f and f_0 in the commutative diagram (26) are trivial, meaning that in the smooth case $\phi = \varphi$.

We finally note that [15, Proposition 1.11] and considerations following Proposition 2.5 in [6] suggest that a similar construction may probably be proposed in the more general setup of Mori Dream Spaces and their ambient toric varieties.

2.5 Maximally bordering chambers and splitting fans. In [2, § 4] Batyrev relates the fibred structure of smooth complete toric varieties with some intersection properties of their primitive collections. In particular, restricting our attention to the subclass of projective varieties, the previous Corollary 2.23, compared with [2, Proposition 4.1], gives that

Given a reduced $n \times (n + r)$ CF-matrix V with $r \geq 2$, a non-singular chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is maximally bordering if and only if there exists a primitive collection $\mathcal{P}$ for Σ_γ such that:

(i) *the corresponding primitive relation $r(\mathcal{P})$ is numerically effective,*
(ii) $\mathcal{P} \cap \mathcal{P}' = \emptyset$ *for any primitive collection $\mathcal{P}'$ for Σ_γ such that $\mathcal{P}' \neq \mathcal{P}$.*

Therefore Theorem 2.22 is clearly the extension of [2, Proposition 4.1] to the case of $\mathbb{Q}$-factorial projective toric varieties.

Moreover the just given characterization of maxbord chambers in terms of primitive collections can be obtained by dropping the smoothness hypothesis too, i.e.

Proposition 2.25. *Given a reduced $n \times (n + r)$ F-matrix V with $r \geq 2$, a chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is maximally bordering if and only if there exists a primitive collection $\mathcal{P}$ for Σ_γ such that:*

(i) *the corresponding primitive relation $r_\mathbb{Z}(\mathcal{P})$ is numerically effective,*
(ii) $\mathcal{P} \cap \mathcal{P}' = \emptyset$ *for any primitive collection $\mathcal{P}'$ for Σ_γ such that $\mathcal{P}' \neq \mathcal{P}$.*

Proof. If γ is maxbord then the existence of a nef primitive collection $\mathcal{P} = \{\langle \mathbf{v}_{s+1}\rangle, \dots, \langle \mathbf{v}_{n+r}\rangle\}$ is guaranteed by Theorem 2.11: in particular we can assume that γ is maxbord with respect to the hyperplane H_r and, saying $\mathcal{P}^* = \{\langle \mathbf{q}_{s+1}\rangle, \dots, \langle \mathbf{q}_{n+r}\rangle\}$, Proposition 2.8 guarantees that $|\sigma(1) \cap \mathcal{P}^*| = 1$ for every cone $\sigma \in \mathcal{B}(\gamma)$. Let $\mathcal{P}'$ be a further primitive collection, for Σ_γ, supported on a hyperplane H' and let $\mathbf{n}'$ be the numerical class of $\mathcal{P}'$, which is the inward primitive normal vector to H'. If there would exist a $\mathbf{q}_i \in \mathcal{P} \cap \mathcal{P}'$, then condition (ii) of

Proposition 2.2 gives a cone $\mathcal{C}_{i,\mathcal{P}'} \in \mathcal{B}(\gamma)$ such that $\mathcal{C}_{i,\mathcal{P}'}(1) \cap \mathcal{P}'^* = \{\langle \mathbf{q}_i \rangle\}$. On the other hand $|\mathcal{C}_{i,\mathcal{P}'}(1) \cap \mathcal{P}^*| = 1$ implying that $\mathcal{C}_{i,\mathcal{P}'}(1) \cap \mathcal{P}^* = \{\langle \mathbf{q}_i \rangle\}$. This is enough to show that $\mathcal{C}_{i,\mathcal{P}'} \cap H' = \mathcal{C}_{i,\mathcal{P}'} \cap H_r$ hence giving $H' = H_r$ and therefore $\mathcal{P}' = \mathcal{P}$.

Conversely let $\mathcal{P} = \{V_P\}$ be a primitive collection satisfying conditions (i) and (ii) and supported by a hyperplane H_P. Then (i) ensures that $\mathcal{P}$ is bordering and Lemma 2.14 allows us to assume that $H_P = H_r$ and $P = \{i \mid s+1 \leq i \leq n+r\}$.

Claim. *Let $\mathbf{q}_i \in \mathcal{P}^*$. If γ is not maxbord with respect to H_r then there exists a hyperplane $H' \neq H_r$, cutting a facet of γ, whose inward normal vector $\mathbf{n}'$ gives $\mathbf{n}' \cdot \mathbf{q}_i > 0$.*

Then the collection supported by H', i.e.

$$\mathcal{P}'^* := \{\mathbf{q}_j \mid \mathbf{q}_j \text{ is a column of } Q \text{ with } \mathbf{n}' \cdot \mathbf{q}_j > 0\}$$

turns out to be a primitive collection such that $\mathcal{P}' \neq \mathcal{P}$ and $\mathcal{P}' \cap \mathcal{P} \supseteq \{\langle \mathbf{q}_i \rangle\} \neq \emptyset$, giving a contradiction.

To prove the Claim we consider all the hyperplanes $H^{(1)}, \dots, H^{(l)}$ cutting a facet of γ. Since γ is not maxbord with respect to H_r, none of them coincides with H_r. Let $\mathbf{n}_j$ be the primitive inward normal vector to $H^{(j)}$. If $\mathbf{n}_j \cdot \mathbf{q}_i \leq 0$ for $1 \leq j \leq l$, then $-\mathbf{q}_i \in \gamma$ giving a contradiction since $-\mathbf{q}_i$ has negative entries. Then there should exist $\mathbf{n}_j$ such that $\mathbf{n}_j \cdot \mathbf{q}_i > 0$. □

With Proposition 2.25 one can then give the following generalization of [2, Theorem 4.3].

Proposition 2.26. *Given a reduced $n \times (n+r)$ CF-matrix V with $r \geq 2$, let $\gamma \in \mathcal{A}_\Gamma(V)$ be a maximally bordering chamber with respect to two distinct hyperplanes H and H'. Then $X(\Sigma_\gamma)$ is a toric cover of a WPTB over a PWS $X'(\Sigma')$ which is still a toric cover of a WPTB.*

Proof. The proof is given by an iterated application of Theorem 2.22. To start the iteration one has to prove that the chamber $\gamma' = \gamma \cap H$, as defined in the proof of the Theorem 2.22, Part (a), possibly up to a toric cover if we are in Cases (b) or (c), is still a maxbord chamber in $\mathrm{Mov}(V')$ with respect to $H \cap H'$, where V' is the fan matrix of the basis of the first, possibly trivial, toric cover. This fact follows by observing that every cone $\sigma \in \mathcal{B}(\gamma)$ has the following properties:

(1) $|\sigma(1) \cap \mathcal{P}^*| = 1$,
(2) $|\sigma(1) \cap \mathcal{P}'^*| = 1$,
(3) $\mathcal{P}^* \cap \mathcal{P}'^* = \emptyset$,

where $\mathcal{P}$ and $\mathcal{P}'$ are nef primitive collections associated with H and H' respectively. Then (1) and (2) follow by the maxbord hypothesis with respect to both these hyperplanes, and (3) follows immediately by Proposition 2.25. Therefore every cone $\sigma \in \mathcal{B}(\gamma)$ admits the decomposition $\sigma = \langle \mathbf{p} \rangle + \langle \mathbf{p}' \rangle + \langle \mathbf{q}_1, \dots, \mathbf{q}_{r-2} \rangle$, with $\mathbf{p} \in \mathcal{P}^*$, $\mathbf{p}' \in \mathcal{P}'^*$ and $\langle \mathbf{q}_1, \dots, \mathbf{q}_{r-2} \rangle \subset H \cap H'$. This suffices to show that γ' is maxbord with respect to $H \cap H'$: in fact all the cones of $\mathcal{B}(\gamma')$ come from a cone in $\mathcal{B}(\gamma)$, since the latter is always the Gale dual of a fibred cone, in the sense of (14). □

We are now in a position of understanding, in the projective case, the concept of a *splitting fan*, as given in Definition 4.2 in [2], in terms of the geometry of the associated chamber. The following definition is crucial.

Definition 2.27. Let V be a reduced $n \times (n+r)$ F-matrix. A chamber $\gamma \in \Gamma(V)$ is called *totally maxbord* if it is maxbord with respect to $r-1$ distinct hyperplanes. Moreover, for $1 \leq l \leq r-1$, the chamber γ is called *l-recursively maxbord* if there exists a sequence of l distinct hyperplanes $H^{(1)}, \dots, H^{(l)}$ such that γ is maxbord with respect to $H^{(1)}$ and, for $0 \leq i \leq l-1$, $\gamma^{(i)} := \gamma \cap \bigcap_{j \leq i} H^{(j)}$ is maxbord with respect to $H^{(i+1)} \cap \bigcap_{j \leq i} H^{(j)}$, possibly up to a finite sequence of toric covers. When $l = r-1$ we simply say that γ is *recursively maxbord*.

Notice that 1-recursively maxbord means simply maxbord. In particular, if $r = 2$ then we have maxbord $\Leftrightarrow$ recursively maxbord $\Leftrightarrow$ totally maxbord.

By an easy induction, the previous Proposition 2.26 shows that *a totally maxbord chamber is a recursively maxbord chamber*. We emphasize that the converse is false, as the following Example 2.41 shows.

We can then state the following generalization of Corollary 4.4 in [2].

Theorem 2.28. *A PWS $X(\Sigma)$ is produced from a toric cover of a WPS by a sequence of toric covers of WPTB's if and only if the corresponding chamber γ_Σ is recursively maxbord.*

The proof is an easy iteration of Theorem 2.22.

Recalling that Batyrev's splitting fan is a non-singular fan whose primitive collections are all disjoint pair by pair, by [2, Corollary 4.4] and the previous Theorem 2.28 a splitting fan turns out to be completely equivalent to a fan associated with a non-singular recursively maxbord chamber. Analogously to what was done in Proposition 2.25 we can try to drop the smoothness hypothesis obtaining the following:

Proposition 2.29. *Given a reduced $n \times (n+r)$ F-matrix V, if $\gamma \in \Gamma(V)$ is a $(r-2)$-recursively maxbord chamber then any two different primitive collection for Σ_γ have no common elements.*

Remark 2.30. In the statement of Proposition 2.29 γ is supposed to be an $(r-2)$-recursively maxbord chamber and not necessarily an $(r-1)$-recursively maxbord one: this fact may result surprising since, after the previous Theorem 2.28, a splitting fan is equivalent to a fan associated with a non-singular recursively maxbord chamber. The following Example 2.31 clarifies the situation, describing a case which cannot occur in the smooth case. Actually, *a non-singular $(r-2)$-recursively maxbord chamber turns out to be necessarily an $(r-1)$-recursively maxbord one.*

This fact is a consequence of [2, Theorem 4.3] and Theorem 2.28. Note that the starting step of the induction proving [2, Theorem 4.3] does no more hold in the singular case, not even for projective varieties; this means that there exist projective $\mathbb{Q}$-factorial toric varieties not admitting any nef primitive collection although their primitive collections are disjoint pair by pair, as Example 2.31 shows.

Example 2.31. Consider the 2-dimensional PWS of rank 2 whose weight and fan matrices are, respectively, given by

$$Q = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \Rightarrow \quad V = \mathcal{G}(Q) = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Then $\mathrm{Mov}(V) = \langle \mathbf{q}_2, \mathbf{q}_3 \rangle = \left\langle \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right\rangle \subset \mathcal{Q} = F^2_+$, and there is a unique chamber $\gamma = \mathrm{Mov}(V)$, giving a unique fan Σ_γ. This fan admits only two disjoint primitive collections given by $\mathcal{P}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{P}_2 = \{\mathbf{v}_3, \mathbf{v}_4\}$ whose primitive relations are given, respectively, by $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_4$ and $\mathbf{v}_3 + 2\mathbf{v}_4 = \mathbf{v}_1$. Hence γ does not admit any nef primitive collection. In particular, note that $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ but γ is not even a bordering chamber.

Proof of Proposition 2.29. We first observe that if $r = 2$ then any chamber always admits only two distinct and disjoint primitive collections.

Assume now that $r \geq 3$ and let γ be a $(r-2)$-recursively maxbord chamber. Then there exists a hyperplane H such that γ is maxbord with respect to H and, possibly up to a toric cover, $\gamma' := \gamma \cap H \in \mathcal{A}_\Gamma(V')$, where V' is a reduced fan matrix of the base of the mentioned toric cover. Let us assume, for ease, that such a toric cover is trivial, hence $V' = \mathcal{G}(Q')$ where Q' is the left-upper $(r-1) \times (n+r-s-1)$ submatrix of $Q = \mathcal{G}(V)$ and it is a W-matrix, as in Part (a) of the proof of Theorem 2.22: such an assumption does not cause any loss of generality since, after a toric cover, the general case is reduced precisely to this situation, as in Cases (b) and (c) of the proof of Theorem 2.22. Let $\mathcal{P}$ be the nef primitive collection associated with H. As a first step we want to show that:

(i) *a primitive collection $\mathcal{P}^{(1)} \neq \mathcal{P}$ for the fan Σ is still a primitive collection for $\Sigma' = \Sigma'_{\gamma'}$.*

We observe that, by the maxbord hypothesis and Proposition 2.25, $\mathcal{P}^{(1)} \cap \mathcal{P} = \emptyset$, meaning that $\mathcal{P}^{(1)*} \subset H$. To prove (i) note that all the rays contained in $\mathcal{P}^{(1)}$ cannot be contained in a unique cone of the fan Σ', which means that, dually, there cannot exist a cone $\sigma' \in \mathcal{B}(\gamma')$ such that $\sigma'(1) \cap \mathcal{P}^{(1)*} = \emptyset$. In fact, by the maxbord hypothesis with respect to H, there exists $\mathbf{p} \notin H$ such that $\sigma = \langle \mathbf{p} \rangle + \sigma' \in \mathcal{B}(\gamma)$: if $\mathcal{P}^{(1)*} \cap \sigma'(1) = \emptyset$ then $\mathcal{P}^{(1)*} \cap \sigma(1) = \emptyset$, since $\mathcal{P}^{(1)*} \subset H$ and $\mathbf{p} \notin H$; this gives a contradiction with the assumption that $\mathcal{P}^{(1)}$ is a primitive collection for Σ.

On the other hand if $\mathbf{p}^{(1)} \in \mathcal{P}^{(1)*}$ then there exists a cone $\sigma \in \mathcal{B}(\gamma)$ such that $\sigma(1) \cap \mathcal{P}^{(1)*} = \{\mathbf{p}^{(1)}\}$. Again the maxbord hypothesis for γ with respect to H gives the existence of $\mathbf{p} \notin H$ and $\sigma' \in \mathcal{B}(\gamma')$ such that $\sigma = \langle \mathbf{p} \rangle + \sigma'$.

Consequently

$$\sigma'(1) \cap \mathcal{P}^{(1)*} = \sigma(1) \cap \mathcal{P}^{(1)*} = \{\mathbf{p}^{(1)}\},$$

since $\mathcal{P}^{(1)} \subset H$ while $\mathbf{p} \notin H$. This suffices to prove (i).

As a second step, observe now that if $\mathcal{P}^{(1)} \neq \mathcal{P}^{(2)}$ are two distinct primitive collections for Σ such that $\mathcal{P}^{(1)} \cap \mathcal{P}^{(2)} \neq \emptyset$ then they give two distinct primitive collections for Σ' admitting common elements. We conclude the proof by induction. □

We now focus on the number of primitive collections. The following result gives a relation between the minimal number of primitive collections and the minimal number of facets of a chamber γ.

Proposition 2.32. *Let V be a reduced $n \times (n+r)$ F-matrix and let $\gamma \in \Gamma(V)$ be a $(r-2)$-recursively maxbord chamber. Then γ is a simplicial cone if and only if the associated fan Σ_γ admits precisely r primitive collections. In particular, if γ is simplicial and*

(1) *either $r \geq 3$*

(2) *or γ is recursively maxbord,*

then at least one primitive collection is numerically effective.

Proof. We start by proving the only if condition: in fact after Theorem 1.4 in [10] one knows that every facet of the chamber γ generates a primitive collection $\mathcal{P}$: e.g. by thinking of $\mathcal{P}^*$ as all the columns $\mathbf{q}$ of $Q = \mathcal{G}(V)$ such that $\mathbf{n} \cdot \mathbf{q} > 0$, where $\mathbf{n}$ is the inward primitive normal vector $\mathbf{n}$ to the considered facet. Then γ admits the minimal number r of facets, meaning that it is necessarily simplicial.

For the converse, we note that when $r = 2$ every chamber γ is simplicial and admits precisely 2 primitive collections. For the second part of the statement note that, in this case, one of these two primitive collections is nef if and only if γ is bordering, hence maxbord.

We now assume $r \geq 3$ and let γ be a simplicial $(r-2)$-recursively maxbord chamber. Let $H^{(1)}$ be a hyperplane with respect to which γ is maxbord: then $H^{(1)}$ cuts out a facet of γ and, possibly up to a toric cover, $\gamma' := \gamma \cap H^{(1)} \in \mathcal{A}_\Gamma(V')$, where V' is a reduced fan matrix of the base of this toric cover. Let $H^{(2)}, \dots, H^{(r)}$ be the further $r-1$ hyperplanes cutting out the remaining facets of γ. For $1 \leq i \leq r$, $H^{(i)}$ is the support of a primitive collection $\mathcal{P}^{(i)}$ defined by setting $\mathcal{P}^{(i)*} = \{\mathbf{q} \in \mathcal{Q}(1) \mid \mathbf{n}_i \cdot \mathbf{q} > 0\}$, where $\mathbf{n}_i$ is the primitive inward normal vector to $H^{(i)}$. By Proposition 2.29, $i \neq j$ implies $\mathcal{P}^{(i)} \cap \mathcal{P}^{(j)} = \emptyset$ and the first step (i) in the proof of this proposition ensures that $\mathcal{P}^{(2)}, \dots, \mathcal{P}^{(r)}$ are $r-1$ distinct primitive collections for $\Sigma' = \Sigma'_{\gamma'}$. Assume now by induction that Σ' admits precisely $r-1$ primitive collections, meaning that $\mathcal{P}^{(2)}, \dots, \mathcal{P}^{(r)}$ give all the primitive collections for Σ'. By the same argument, if $\mathcal{P} \neq \mathcal{P}^{(1)}$ is a primitive collection of Σ then it is a primitive collection of Σ', which means, by induction, that $\mathcal{P} = \mathcal{P}^{(i)}$ for some i with $2 \leq i \leq r$. This suffices to show that Σ admits precisely r primitive collections given by the facets of γ. Now γ is maxbord with respect to $H^{(1)}$, hence $\mathcal{P}^{(1)}$ is nef. □

We now assume that $r \leq 3$. The previous Proposition 2.32 allows us to prove the following result extending to the singular case an analogous result proven by Batyrev in Sections 5 and 6 of [2] under a smoothness hypothesis (see the following Remark 2.34).

Theorem 2.33. *Let V be a reduced $n \times (n+r)$ F-matrix with $r \leq 3$. If $\gamma \in \Gamma(V)$ is a maxbord chamber then γ is simplicial and the associated fan Σ_γ admits precisely r primitive collections and at least one of them is numerically effective.*

Proof. Let γ be a maxbord chamber with respect to the hyperplane H. Note that

- *if $r \leq 3$ then a maxbord chamber is a simplicial cone.*

In fact, if $r \leq 2$ there is nothing to prove since every chamber is simplicial. Let us assume $r = 3$. Then every cone $\sigma \in \mathcal{B}(\gamma)$ can be written as follows:

$$\sigma = \langle \mathbf{p} \rangle + \sigma', \quad \text{with } \mathbf{p} \notin H, \sigma' \subset H \tag{27}$$

Because $\gamma = \bigcap_{\sigma \in \mathcal{B}(\gamma)} \sigma$, (27) shows that γ admits a unique ray outside of the hyperplane H, generated by some $\mathbf{p} \notin H$. Hence $\gamma = \mathbf{p} + \sigma'$ for some $\sigma' \subset H$ which is necessarily simplicial since $\dim(\sigma') = 2$. Then

Proposition 2.32 concludes the proof. In particular the primitive collection associated with the hyperplane H is nef. □

Remark 2.34. In view of Remark 2.30, the previous Theorem 2.33 generalizes a result already proved by Batyrev under the further hypothesis that γ is non-singular, meaning that Σ_γ is a splitting fan in the sense of Definition 4.2 in [2] (see Propositions 5.2–5, Theorem 5.7 and Theorem 6.6 in [2]). Actually Batyrev proved also the converse result, which is: a non-singular fan admitting precisely 3 primitive collections is necessarily a splitting fan, which means, by Corollary 4.4 in [2] and Theorem 2.28, that γ is a non-singular recursively maxbord chamber. Note that this latter fact is false in the singular case, as Example 2.44 shows.

2.6 Maximally bordering chambers and contractible primitive relations. We conclude the present subsection by giving a partial generalization of results by Sato and Casagrande. We first recall the following definition.

Definition 2.35 ([5], Definition 2.3). A class $\kappa \in A_1(X) \cap \overline{\mathrm{NE}}(X)$ is called *contractible* if κ is a generator of the semigroup $A_1(X) \cap \mathbb{Q}_{\geq 0}\kappa$ and there exist

- some irreducible curve having numerical class in $\mathbb{Q}_{\geq 0}\kappa$,
- a toric variety X_κ,
- an equivariant morphism $\varphi_\kappa : X \longrightarrow X_\kappa$ with connected fibers,

such that for every irreducible curve $C \subset X$

$$\varphi_\kappa(C) = \{pt\} \iff [C] \in \mathbb{Q}_{\geq 0}\kappa.$$

We now assume that X is smooth: then $r_{\mathbb{Z}}(\mathcal{P}) = r(\mathcal{P})$. Corollary 2.4 and Proposition 3.4 in [5] jointly with Proposition 2.25 above allow one to conclude the following.

Proposition 2.36. *Let $X(\Sigma)$ be a smooth projective toric variety. Then the following facts are equivalent:*

(1) *there exists a numerically effective primitive relation $\kappa = r(\mathcal{P})$ for Σ which is contractible,*
(2) *there exists a nef primitive collection $\mathcal{P}$ for Σ such that for every primitive collection $\mathcal{P}' \neq \mathcal{P}$ for Σ, one has $\mathcal{P}' \cap \mathcal{P} = \emptyset$,*
(3) *γ_Σ is a maxbord chamber.*

In particular X_κ is smooth of dimension $n - s$ and rank $r - 1$ and φ_κ is a toric $\mathbb{P}^s$-bundle.

Proof. (1) ⇔ (2). This is precisely [5, Proposition 3.4]. In particular the contractible primitive relation κ in Part (1) is the primitive relation $\kappa = r(\mathcal{P})$ of a primitive collection $\mathcal{P}$ like in Part (2) and viceversa.

(2) ⇔ (3). This is Proposition 2.25. In particular γ_Σ turns out to be maxbord with respect to the the hyperplane $H_{\mathcal{P}}$ supporting the primitive collection $\mathcal{P}$ as in Part (2) and viceversa.

Finally [5, Corollary 2.4] gives the last part of the statement. □

Theorem 2.22 and Proposition 2.25 allow us to extend the previous result to a PWS, although the situation turns out to be more intricate. First we note that if $f : X(\Sigma) \to \widetilde{X}(\widetilde{\Sigma})$ is a toric cover, then Lemma 2.21 guarantees that $\gamma = \gamma_\Sigma$ is a maxbord chamber if and only if $\widetilde{\gamma} = \widetilde{\gamma}_{\widetilde{\Sigma}}$ is a maxbord chamber. Let H and $\widetilde{H}$ be bordering hyperplanes of γ and $\widetilde{\gamma}$, respectively, and let $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ be the collections supported by H and $\widetilde{H}$, respectively, and such that $\widetilde{\mathcal{P}} = \{f_{\mathbb{R}}(\mathbf{v}) \mid \mathbf{v} \in \mathcal{P}\}$: in this case we will write $\widetilde{\mathcal{P}} = f(\mathcal{P})$. Proposition 2.25 implies that $\mathcal{P}$ and $\widetilde{\mathcal{P}} = f(\mathcal{P})$ are nef primitive collections for Σ and $\widetilde{\Sigma}$, respectively, both satisfying property (2) in Proposition 2.36. Consider the associated numerically effective primitive relations $\kappa = r_{\mathbb{Z}}(\mathcal{P})$ and $\widetilde{\kappa} = r_{\mathbb{Z}}(\widetilde{\mathcal{P}})$: also in this case we write $\widetilde{\kappa} = f(\kappa)$.

Definition 2.37. Given a toric cover $f : X(\Sigma) \to \widetilde{X}(\widetilde{\Sigma})$, a numerically effective primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P}) \in A_1(X) \cap \overline{\mathrm{NE}}(X)$ for Σ is called *pseudo-contractible* if $\widetilde{\kappa} = f(\kappa)$ is a contractible class in $A_1(\widetilde{X}) \cap \overline{\mathrm{NE}}(\widetilde{X})$. Then there exist a toric variety $X_{\widetilde{\kappa}}$ and the following commutative diagram of equivariant morphisms

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi_\kappa"'] & \widetilde{X} \arrow[d, "\varphi_{\widetilde{\kappa}}"] \\ & X_{\widetilde{\kappa}} \end{tikzcd} \tag{28}$$

such that $\varphi_{\widetilde{\kappa}}$ has connected fibers and for every irreducible curve $C \subset X$

$$\varphi_\kappa(C) = \{pt\} \iff [f(C)] \in \mathbb{Q}_{\geq 0}\widetilde{\kappa}.$$

Theorem 2.38. *Let V be a reduced $n \times (n+r)$ CF-matrix and $\Sigma \in \mathbb{P}\mathcal{SF}(V)$. Assume that there exists a primitive collection $\mathcal{P}$ for Σ whose primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P})$ is numerically effective. By applying Lemma 2.14, assume that $H_r = \{x_r = 0\} \subseteq F^r_{\mathbb{R}}$ is the supporting hyperplane of $\mathcal{P}$, meaning that there exists a W-matrix $Q = \mathcal{G}(V)$ in positive* REF *and looking as in* (16), *which is*

$$Q = \begin{pmatrix} \overbrace{\begin{matrix} Q' \\ 0 \cdots 0 \end{matrix}}^{n'+r'} & \overbrace{\begin{matrix} Q'' \\ w_0 \cdots w_s \end{matrix}}^{s+1} \end{pmatrix}$$

hence giving $r_{\mathbb{Z}}(\mathcal{P}) = (w_0, \dots, w_s)$. Then, setting $V' = \mathcal{G}(Q')$:

(1) *κ is contractible if and only if the chamber γ_Σ is maxbord with respect to the hyperplane H_r and*
   (1.i) *Q' is a $(r-1) \times (n+r-s-1)$ reduced W-matrix,*
   (1.ii) *the columns of Q'' are classes of $s+1$ Cartier divisors of $X'(\Sigma')$ where $\Sigma' \in \mathbb{P}\mathcal{SF}(V')$ is the fan associated with the chamber $\gamma' = \gamma \cap H_r$,*
   In particular, the contraction $\varphi_\kappa : X(\Sigma) \longrightarrow X_\kappa = X'(\Sigma')$ exhibits X as WPTB whose fibers are isomorphic to the s-dimensional WPS $\mathbb{P}(w_0, \dots, w_s)$.

(2) *κ is pseudo-contractible if and only if γ_Σ is maxbord with respect to the hyperplane H_r and either* (1.*i*) *holds and*
   (2.ii) *there exists a column of Q'' giving the class of a Weil non-Cartier divisor of $X'(\Sigma')$ where $\Sigma' \in \mathbb{P}\mathcal{SF}(V')$ is the fan associated with the chamber $\gamma' = \gamma \cap H_r$,*
   or
   (2.i) *either Q' is a $(r-1) \times (n+r-s-1)$ non-reduced W-matrix or Q' satisfies all the conditions of Definition* 1.3 *but condition b.*
   In particular, in the former case $\varphi_\kappa : X(\Sigma) \to X'(\Sigma')$ exhibits X as a WPTwB and in any case X is a toric cover of a WPTB.

In any case, either (1) *or* (2) *occurs if and only if one of the following equivalent conditions happens:*

(I) *γ_Σ is a maxbord chamber,*
(II) *for every primitive collection $\mathcal{P}' \neq \mathcal{P}$ for Σ, one has $\mathcal{P}' \cap \mathcal{P} = \emptyset$.*

Proof. Case (1) is an application of Theorem 2.22 in the easiest situation in which Q is a W-matrix of a WPTB. Then techniques proving [5, Proposition 3.4] and [27, Theorem 1.10] here apply as in the smooth case. Proving Case (2) reduces to Case (1) after we consider the toric covers described in 2.2.3, to settle the Case (2.ii), and in Parts (b) and (c) of the proof of Theorem 2.22, to settle the remaining Case (2.i). Finally the equivalence of conditions (I) and (II) follows immediately by Proposition 2.25. □

Remark 2.39. Let us here recall and apply what has been observed in § 2.4. The fact that the chamber γ_Σ is maxbord with respect to the hyperplane H actually implies that the primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P}_H)$, supported by H, is a nef contractible class of $A_1(X) \cap \overline{\mathrm{NE}}(X)$, whose contraction gives a fibering morphism $\phi : X(\Sigma) \to X_0(\Sigma_0)$. Hence

a pseudo-contractible class is actually a contractible class.

More precisely the fibering morphism ϕ turns out to be the morphism with connected fibers of the Stein factorization of the composition $\varphi_\kappa = \varphi_{\widetilde{\kappa}} \circ f$ in the commutative diagram (28), hence giving the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f", "\text{toric cover}"'] \arrow[d, "\phi"] \arrow[dr, "\varphi_\kappa"] & \widetilde{X} \arrow[d, "\varphi_{\widetilde{\kappa}}"] \\ X_0 \arrow[r, "f_0", "\text{finite morphism}"'] & X_{\widetilde{\kappa}} \end{tikzcd} \tag{29}$$

The geometric description of the Stein factorization $f_0 \circ \phi$ of φ_κ on the left hand side of diagram (29) is well known and has its roots in Reid's paper [20] (see also [9, Lemma 15.4.2 and Proposition 15.4.5] and references therein): nevertheless it simply says that the fibers of ϕ are given by a fake WPS.

By Theorem 2.38, the factorization $\varphi_{\widetilde{\kappa}} \circ f$ of φ_κ on right hand side of diagram (29) is completely described, starting from a fan matrix V of X, in terms of all the matrices representing morphisms giving diagram (25).

2.7 Examples. In this section we give some applications of techniques just illustrated. We start with the case of smooth projective toric varieties presented along with the introduction of definitions and constructions given above, to compare Batyrev's techniques described in [2] with techniques presented here.

Example 2.40 (Examples 1.5, 1.12 and 2.13 continued). Consider the smooth and projective toric variety $X(\Sigma)$, of dimension and rank equal to 3, with $\Sigma \in \mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V) = \{\Sigma_1, \Sigma_2\}$ defined in the Example 1.5. Recall Figure 1, to visualize the Gale dual cone $\mathcal{Q} = \langle Q\rangle = F^3_+$ and $\mathrm{Mov}(V) \subseteq \mathcal{Q}$ and the two chambers γ_1, γ_2, explicitly presented in Example 1.12.

In Example 2.13 we observed that both chambers are intbord with respect to both the hyperplanes H_2 and H_3 and moreover γ_1 is maxbord with respect to these hyperplanes: hence it is totally maxbord. Hyperplanes H_2 and H_3 are supporting collections, $\mathcal{P}_2 = \{\mathbf{v}_3, \mathbf{v}_4\}$ and $\mathcal{P}_3 = \{\mathbf{v}_5, \mathbf{v}_6\}$, respectively, which are primitive and nef for both the fans Σ_1 and Σ_2. Batyrev's results [2, Proposition 4.1, Theorem 4.3, Corollary 4.4], or equivalently Corollary 2.23 and Theorem 2.28 above, imply that $X(\Sigma_1)$ is a PTB over a smooth toric surface of rank 2.

The weight matrix Q makes this fact quite explicit: the last row of Q gives the class $\kappa := r(\mathcal{P}_3) \in A_1(X(\Sigma_i)) \cap \overline{\mathrm{NE}}(X(\Sigma_i))$ which is a numerically effective primitive relation for both the fans Σ_1, Σ_2. The class κ turns out to be contractible when the hypotheses of [5, Proposition 3.4] are verified: more easily κ turns out to be contractible by Condition (3) in Proposition 2.36 above. The contraction morphism ϕ_κ is now explicitly described by the weight matrix Q, following Corollary 2.23: namely $\varphi_\kappa : X(\Sigma_1) \longrightarrow Y(\Sigma')$ exhibits $X(\Sigma_1)$ as a PTB, whose fibers are isomorphic to $\mathbb{P}^1$, over the toric surface Y, whose weight matrix Q' is obtained from Q by deleting the third row and the 5-th and 6-th columns, and whose fan Σ' is the unique simplicial fan in $\mathcal{SF}(\mathcal{G}(Q'))$, which is

$$Q' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \gamma' = \gamma_{\Sigma'} = \left\langle \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right\rangle \quad \Longrightarrow \quad Y \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big).$$

By subtracting the third row and the second row from the first one in Q, and recalling the role of the matrix Q'' in (16), one gets the following weight matrix of $X(\Sigma_1)$

$$Q \sim \widetilde{Q} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad X(\Sigma_1) \cong \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(h))$$

where $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and h is the generator of $\mathrm{Pic}(Y)$ given by the pull-back of the Picard generator $\mathcal{O}_{\mathbb{P}^1}(1)$ of the base $\mathbb{P}^1$. Then we get a recursive PTB structure given by

$$X(\Sigma_1) \cong \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(h)) \twoheadrightarrow Y \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big) \twoheadrightarrow \mathbb{P}^1 .$$

Such a fibration of $X(\Sigma_1)$ is not unique since the contraction of the other numerically effective primitive relation $\kappa' := r(\mathcal{P}_2)$ gives precisely the same description of $X(\Sigma_1)$. This fact can be immediately deduced from the weight matrix Q, whose second row gives the class κ': in fact by exchanging the second and the third row and reordering the columns to still get a REF matrix, one still obtains the W-matrix Q.

$X(\Sigma_2)$ is now obtained by $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(h))$ after the elementary flip determined by crossing the internal wall $\langle \mathbf{q}_3, \mathbf{q}_5\rangle$ of $\mathrm{Mov}(V)$. Therefore the indetermination loci of the elementary flip $X(\Sigma_1) \dashleftarrow\dashrightarrow X(\Sigma_2)$ are

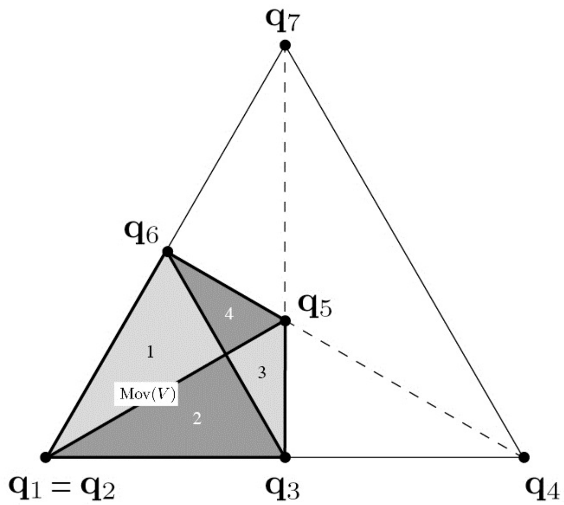

Figure 2: Example 2.41: the section of the cone Mov(V) and its four chambers, inside the Gale dual cone $\mathcal{Q} = F^3_+$, as cut out by the plane $\sum_{i=1}^3 x_i^2 = 1$.

given by invariant 1-cycles $C_1 := O(\langle \mathbf{v}_4, \mathbf{v}_6\rangle) \subset X(\Sigma_1)$ and $C_2 := O(\langle \mathbf{v}_1, \mathbf{v}_2\rangle) \subset X(\Sigma_2)$. In fact in this way the primitive collections $\mathcal{P} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{P}' = \{\mathbf{v}_4, \mathbf{v}_6\}$, supported by the hyperplane cutting the internal wall, and their foci are exchanged with each other. This means that $\mathcal{P}$ is a primitive collection for Σ_1 and $\mathcal{P}'$ is a primitive collection for Σ_2. Note that $\mathcal{P} \cap \mathcal{P}_3 = \emptyset$, but $\mathcal{P}' \cap \mathcal{P}_3 = \{\mathbf{v}_6\} \neq \emptyset$ and $\mathcal{P}' \setminus \mathcal{P}_3 = \{\mathbf{v}_4\}$, meaning that $X(\Sigma_2)$ is not a toric $\mathbb{P}^1$-bundle over Y and κ is not a contractible class for $X(\Sigma_2)$ (by Proposition 2.36 and [5, Proposition 3.4], respectively).

We finally note that Σ_1 is a splitting fan, by Proposition 2.29. This is clearly not the case for the fan Σ_2: moreover Σ_1 turns out to admit 3 primitive collections and Σ_2 to admit 5 primitive collections, according to [2, § 5].

The following is still an example of smooth projective toric varieties coming from chambers which are recursively maxbord but not maxbord.

Example 2.41. By adding the further column $\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$ in the weight matrix of the previous Example 2.40, one gets the following reduced weight and fan matrices

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Longrightarrow V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix} = \mathcal{G}(Q)$$

The new weight column introduces a further subdivision in $\mathcal{Q} = F^3_+$ along the hyperplane $H : x_2 - x_3 = 0$ through $\mathbf{q}_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) = \mathbf{q}_2$ and $\mathbf{q}_5 = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$, leaving unchanged Mov($V$) and giving $\mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V) = \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$. See Figure 2. The four simplicial complete fans Σ_i are described by the following chambers

$$\gamma_1 = \langle \mathbf{q}_1 = \mathbf{q}_2, \mathbf{w}, \mathbf{q}_6\rangle = \left\langle \begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \right\rangle, \quad \gamma_2 = \langle \mathbf{q}_1 = \mathbf{q}_2, \mathbf{q}_3, \mathbf{w}\rangle = \left\langle \begin{matrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\rangle$$

$$\gamma_3 = \langle \mathbf{q}_3, \mathbf{w}, \mathbf{q}_5\rangle = \left\langle \begin{matrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \right\rangle, \quad \gamma_4 = \langle \mathbf{w}, \mathbf{q}_5, \mathbf{q}_6\rangle = \left\langle \begin{matrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \right\rangle$$

respectively, where $\mathbf{q}_1, \dots, \mathbf{q}_7$ are the columns of Q and $\mathbf{w} := \mathbf{q}_3 + \mathbf{q}_6 = \mathbf{q}_1 + \mathbf{q}_5 = \mathbf{q}_2 + \mathbf{q}_5$. We observe that $X(\Sigma_i)$ is smooth for every $i = 1, \dots, 4$.

In particular both γ_1 and γ_2 are recursively maxbord chambers, γ_1 with respect to the sequence of hyperplanes H_2, H and γ_2 with respect to the sequence of hyperplanes H_3, H: note that none of them is totally maxbord.

We first describe the sequence of PTB's describing $X(\Sigma_2)$ and given by the recursively maxbord structure of γ_2. The last row of Q gives the numerically effective primitive relation $\kappa = r(\mathcal{P}_2)$ of the primitive collection $\mathcal{P}_2 = \{\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$, for Σ_2, associated with the maxbord hyperplane H_3. The left-upper 2×4 submatrix Q' with respect to the primitive collection $\mathcal{P}_2^*$ is the same as in the previous Example 2.40. Therefore the contraction of κ gives the morphism $\varphi_\kappa : X(\Sigma_2) \longrightarrow Y(\Sigma')$, where Y has weight matrix Q' and Σ' is the unique simplicial fan in $\mathcal{SF}(\mathcal{G}(Q'))$, associated with the chamber $\gamma_2' = \gamma_2 \cap H_3 = \langle \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \rangle$.

Note that $H_3 \cap H$ gives the line generated by $\mathbf{q}_1$, hence the hyperplane H_2' of $\mathcal{Q}'$ with respect to γ_2' is clearly maxbord. Hence, as above, $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and the contraction of the primitive relation $\mathcal{P}_2'$, associated with γ_2', gives the structural projection $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \twoheadrightarrow \mathbb{P}^1$. On the other hand, by first subtracting the second row from the first one and then subtracting the third row from the previous ones in Q, one gets the following weight matrix of $X(\Sigma_i)$

$$Q \sim \widetilde{Q} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

which gives

$$X(\Sigma_2) \cong \mathbb{P}(\mathcal{O}_Y(h) \oplus \mathcal{O}_Y(f) \oplus \mathcal{O}_Y(f+h))$$

where f, h are the generators of $\mathrm{Pic}(Y)$ given by the fibre and the pull-back of the Picard generator of the base $\mathbb{P}^1$, respectively. Therefore $X(\Sigma_2)$ is obtained from $\mathbb{P}^1$ by the following sequence of PTB's:

$$X(\Sigma_2) \cong \mathbb{P}(\mathcal{O}_Y(h) \oplus \mathcal{O}_Y(f) \oplus \mathcal{O}_Y(f+h)) \twoheadrightarrow Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \twoheadrightarrow \mathbb{P}^1 .$$

For what concerns $X(\Sigma_1)$, by exchanging the second and the third rows in Q and reordering the columns one gets still the same weight matrix, but now the last row gives the primitive relation $\kappa' = r(\mathcal{P}_1)$ of the primitive collection $\mathcal{P}_1 = \{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ for Σ_1. By the previous analysis we still get

$$X(\Sigma_1) \cong \mathbb{P}(\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(f') \oplus \mathcal{O}_{Y'}(f'+h')) \twoheadrightarrow Y' \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \twoheadrightarrow \mathbb{P}^1 .$$

The elementary flip $X(\Sigma_1) \dashleftarrow\dashrightarrow X(\Sigma_2)$ is then obtained by crossing the internal wall of $\mathrm{Mov}(V)$ cut out by the hyperplane H and the indetermination loci are described by the foci of the primitive collections supported by H with respect to the two fans Σ_1 and Σ_2, namely $\mathcal{P}_1' = \{\mathbf{v}_6, \mathbf{v}_7\}$ for Σ_1 and $\mathcal{P}_2' = \{\mathbf{v}_3, \mathbf{v}_4\}$ for Σ_2, whose foci are given by the cones $\langle \mathbf{v}_3, \mathbf{v}_4 \rangle$ and $\langle \mathbf{v}_6, \mathbf{v}_7 \rangle$, respectively. The indetermination loci are then given by $C_1 = O(\langle \mathbf{v}_3, \mathbf{v}_4 \rangle) \subset X(\Sigma_1)$ and $C_2 = O(\langle \mathbf{v}_6, \mathbf{v}_7 \rangle) \subset X(\Sigma_2)$.

We finally observe that both the chambers γ_1 and γ_2 are simplicial, accordingly with Proposition 2.32: moreover they both admit three primitive collections, namely

$$\begin{aligned} \Sigma_1 \rightsquigarrow \mathcal{P}_1 = \{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}, \quad \mathcal{P}_1' = \{\mathbf{v}_6, \mathbf{v}_7\}, \quad \mathcal{P}'' = \{\mathbf{v}_1, \mathbf{v}_2\} \\ \Sigma_2 \rightsquigarrow \mathcal{P}_2 = \{\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}, \quad \mathcal{P}_2' = \{\mathbf{v}_3, \mathbf{v}_4\}, \quad \mathcal{P}'' = \{\mathbf{v}_1, \mathbf{v}_2\}. \end{aligned}$$

In particular both Σ_1 and Σ_2 are splitting fans, while this clearly does not hold for Σ_3 and Σ_4.

The following examples deal with singular $\mathbb{Q}$-factorial PWS, applying techniques introduced in this paper. We start by giving an example not satisfying the hypothesis of Theorem 2.24.

Example 2.42. Consider the 2-dimensional PWS $X(\Sigma)$, of rank 3, whose reduced fan matrix is given by

$$V = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -2 & 1 & -1 \end{pmatrix} \quad \Longrightarrow \quad Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \mathcal{G}(V).$$

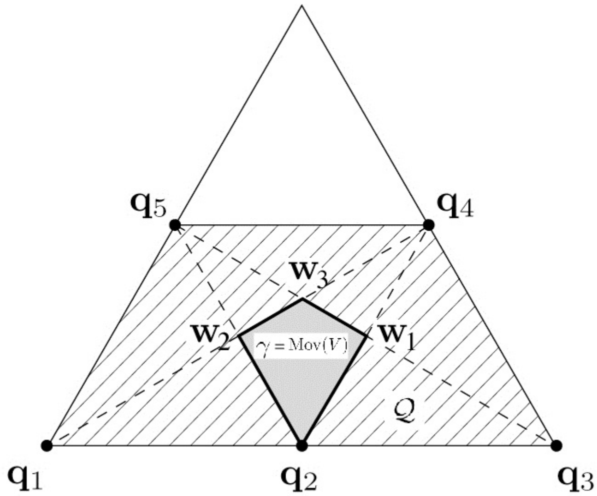

Figure 3: Example 2.42: the section of the cone $\gamma = \mathrm{Mov}(V)$ inside the Gale dual cone $\mathcal{Q} \subset F^3_+$, as cut out by the plane $\sum_{i=1}^3 x_i^2 = 1$.

Then $\mathbb{P}\mathcal{SF}(V) = \mathcal{SF}(V) = \{\Sigma\}$ and the unique simplicial complete fan Σ is associated with the unique chamber of $\mathrm{Mov}(V) \subset \mathcal{Q}$

$$\gamma = \mathrm{Mov}(V) = \langle \mathbf{q}_2, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \rangle = \left\langle \begin{matrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{matrix} \right\rangle$$

where $\mathbf{q}_1, \dots, \mathbf{q}_5$ are the columns of Q and

$$\mathbf{w}_1 := \mathbf{q}_2 + \mathbf{q}_4, \quad \mathbf{w}_2 = \mathbf{q}_2 + \mathbf{q}_5, \quad \mathbf{w}_3 = \mathbf{q}_1 + \mathbf{q}_4 = \mathbf{q}_3 + \mathbf{q}_5$$

(see Figure 3). Note that the cone $\sigma = \left\langle \begin{smallmatrix} 1 & -1 \\ 0 & -2 \end{smallmatrix} \right\rangle$ is a maximal cone of Σ, hence $X(\Sigma)$ turns out to admit a singular point. The last row of Q gives the numerical effective primitive relation $r_{\mathbb{Z}}(P) \in A_1(X(\Sigma)) \cap \overline{\mathrm{NE}}(X(\Sigma))$ of the primitive collection $\mathcal{P} = \{\mathbf{v}_4, \mathbf{v}_5\}$. Moreover γ is not a maxbord chamber and, being the unique chamber in $\mathrm{Mov}(V)$, the PWS X cannot be birational and isomorphic in codimension 1 to a toric cover of a WPTB, by Theorem 2.22. Note that, by Theorem 2.24, this fact is equivalent to asserting that the left-upper 2×3 submatrix Q' of Q, with respect to the primitive collection $\mathcal{P}$, has to violate one of the conditions in Definition 1.3 different from Condition (b). In fact $Q' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ giving that $(1, 0, -1) \in \mathcal{L}_r(Q)$, contradicting the Condition (f) in the Definition 1.3.

Example 2.43. The present example gives an account of the Case (b) in the proof of Theorem 2.22. Consider a 4-dimensional PWS of rank 3 given by the following reduced fan and weight matrices

$$V = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} = \mathcal{G}(V)$$

The first interest of this example is in the fact that $|\mathbb{P}\mathcal{SF}(V)| = 8 < 10 = |\mathcal{SF}(V)|$, meaning that V carries two distinct fans of $\mathbb{Q}$-factorial complete toric varieties of rank 3 which are not projective. In particular, $X(\Sigma)$ is singular for every fan $\Sigma \in \mathcal{SF}(V)$. Figure 4 describes the Gale dual cone $\mathcal{Q} = \langle Q \rangle$ and $\mathrm{Mov}(V)$ with its eight chambers γ_i where $i = 1, \dots, 8$. Following the notation introduced in Example 2.40, the associated fans $\Sigma_i := \Sigma_{\gamma_i}$ are given by all the faces of their maximal cones. The two non-projective fans are generated by the

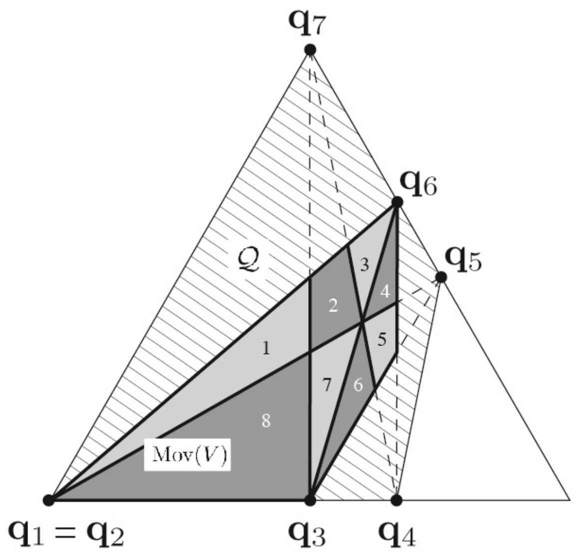

Figure 4: Example 2.43: this is the section cut out by the plane $\sum x_i^2 = 1$ of the cone Mov(V), with its eight chambers, inside the Gale dual cone $\mathcal{Q} \subset F_+^3$.

following list of maximal cones:

$$\begin{aligned}\Sigma_9(4) = \{&\{2,3,4,7\},\{2,3,4,6\},\{2,3,5,7\},\{2,3,5,6\},\{1,3,4,7\},\{1,2,4,7\},\\&\{1,3,4,6\},\{1,2,4,6\},\{1,3,5,7\},\{1,2,5,7\},\{1,3,5,6\},\{1,2,5,6\}\}\\ \Sigma_{10}(4) = \{&\{2,4,5,7\},\{2,4,6,7\},\{2,3,5,7\},\{2,3,6,7\},\{1,4,5,7\},\{1,4,6,7\},\\&\{1,2,4,5\},\{1,2,4,6\},\{1,3,5,7\},\{1,2,3,5\},\{1,2,3,6\},\{1,3,6,7\}\}.\end{aligned}$$

The intersection of the cones in the associated bunches of cones inside $\mathcal{Q}$, gives, in both cases, the 1-dimensional cone

$$\langle \mathbf{w}_1\rangle \quad \text{with} \quad \mathbf{w}_1 = \mathbf{q}_3 + \mathbf{q}_6 = \mathbf{q}_1 + 2\mathbf{q}_5 = \mathbf{q}_2 + 2\mathbf{q}_5 = \mathbf{q}_4 + 2\mathbf{q}_7 = \begin{pmatrix}1\\2\\2\end{pmatrix}$$

which is the primitive generator of the common ray to the six chambers γ_i where $2 \le i \le 7$ (see Figure 4). Among the 8 distinct chambers of Mov(V) giving the projective fans, the unique maxbord chamber is given by

$$\gamma_8 = \langle \mathbf{q}_1 = \mathbf{q}_2, \mathbf{q}_3, \mathbf{w}\rangle, \quad \text{with} \quad \mathbf{w} = \mathbf{q}_3 + \mathbf{q}_7 = \mathbf{q}_1 + \mathbf{q}_5 = \mathbf{q}_2 + \mathbf{q}_5 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

which actually is also a recursively (no totally) maxbord chamber with respect to the sequence of hyperplanes H_3, H, where H is the hyperplane $x_2 - x_3 = 0$, through $\mathbf{q}_1$ (or, equivalently, $\mathbf{q}_2$) and $\mathbf{q}_5$. The maxbord hyperplane H_3 supports the nef primitive collection $\mathcal{P} = \{\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$ whose numerically effective primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P})$ is given by the last row of Q. Note that the left-upper 2×4 submatrix $Q' = \begin{pmatrix}1&1&1&1\\0&0&1&2\end{pmatrix}$ of Q is a non-reduced W-matrix: in fact $\mathcal{L}_r(Q^{\{3\}}) \subset \mathbb{Z}^3$ has cotorsion, admitting the generator $(0,0,2)$. We are then in Case (b) in the proof of Theorem 2.22. Using the same notation therein, we then get

$$A = \operatorname{diag}(1, 1/2, 1/2) \in \mathrm{GL}_3(\mathbb{Q}), \qquad B = \operatorname{diag}(1,1,2,1,2,2,2) \in \mathrm{GL}_7(\mathbb{Q}) \cap \mathbf{M}_7(\mathbb{Z}) \tag{30}$$

which give

$$\widetilde{Q} = AQB = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \qquad \widetilde{V} = \mathcal{G}(\widetilde{Q}) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

Recall that V is a CF-matrix, then $H := \mathrm{HNF}(V^T) = \begin{pmatrix} I_4 \\ \mathbf{0}_{3,4} \end{pmatrix}$ by [23, Theorem 2.1(4)]. Let $U \in \mathrm{GL}_7(\mathbb{Z})$ such that $U \cdot V^T = H$. Then the upper 4 rows of U give ${}^4U \cdot V^T = I_4$ (recall notation in list 1.1). Therefore

$$V^T \cdot C = B \cdot \widetilde{V}^T \quad \Longrightarrow \quad C = {}^4U \cdot B \cdot \widetilde{V}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since $\det C = 4$, denoting by $\overline{f} : N \to \widetilde{N}$ the map represented by C^T, one gets that $\overline{f}(N)$ is a subgroup of index 4 in $\widetilde{N}$. The fan $\widetilde{\Sigma} = \overline{f}_{\mathbb{R}}(\Sigma_8)$ is then the fan defined by the simplicial chamber

$$\widetilde{\gamma} := \langle \widetilde{\mathbf{q}}_1 = \widetilde{\mathbf{q}}_2, \widetilde{\mathbf{q}}_3, \widetilde{\mathbf{w}} \rangle = \left\langle \begin{matrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\rangle \subset \mathrm{Mov}(\widetilde{V}) \tag{31}$$

where $\widetilde{\mathbf{q}}_i$ are the columns of $\widetilde{Q}$ and $\widetilde{\mathbf{w}} = \widetilde{\mathbf{q}}_3 + \widetilde{\mathbf{q}}_7 = \left(\begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix}\right)$. It is still a recursively maxbord chamber with respect to the same sequence of hyperplanes H_3, H. The maxbord hyperplane H_3 supports the nef primitive collection $\widetilde{\mathcal{P}} = \{\widetilde{\mathbf{v}}_5, \widetilde{\mathbf{v}}_6, \widetilde{\mathbf{v}}_7\}$ whose numerically effective primitive relation $\widetilde{\kappa} = r_{\mathbb{Z}}(\widetilde{\mathcal{P}})$ is given by the last row of $\widetilde{Q}$. Now the left-upper 2×4 submatrix $\widetilde{Q}' = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ of $\widetilde{Q}$ turns out to be a reduced W-matrix, hence we are now in Case (a) of the proof of Theorem 2.22 and $\widetilde{X}(\widetilde{\Sigma})$ turns out to be either a WPTwB or a WPTB.

We first describe the toric cover $f : X \to \widetilde{X}$ induced by the homomorphism $\overline{f} : N \cong \mathbb{Z}^n \to \widetilde{N} \cong \mathbb{Z}^n$, represented by the transposed matrix C^T. Recalling the Cox description of X and $\widetilde{X}$ as geometric quotients, the covering f is then completely described by taking the non-trivial entries of the diagonal matrix B^T as exponents of the Cox ring variables of X to obtain the Cox ring variables of $\widetilde{X}$. One then gets that

f is a double covering ramified along the torus-invariant divisors D_3, D_5, D_6 and D_7.

On the other hand, to describe the structure of $\widetilde{X}$, note that $\widetilde{Q}' \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Hence, proceeding as in Example 2.40 and applying Theorem 2.38, one gets that $\widetilde{X}$ is a WPTB over the PTB $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, whose weights are given by $W = (1, 2, 1)$. The fibration is given by the contraction $\varphi_{\widetilde{\kappa}}$ of the class $\widetilde{\kappa}$. After suitable operations on the rows of $\widetilde{Q}$, one gets the following weight matrix of $\widetilde{X}$

$$\widetilde{Q} \sim \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

hence giving

$$\widetilde{X}(\widetilde{\Sigma}) \cong \mathbb{P}^{(1,2,1)}(\mathcal{O}_Y(2h) \oplus \mathcal{O}_Y(f + 2h) \oplus \mathcal{O}_Y(f))$$

where f, h are the generators of $\mathrm{Pic}(Y)$ given by the fibre and the pull-back of the Picard generator $\mathcal{O}_{\mathbb{P}^1}(1)$ of the base $\mathbb{P}^1$, respectively.

In conclusion, $X(\Sigma_8)$ *is obtained from* $\mathbb{P}^1$ *by means of the following sequence of toric covers and WPTB's*

$$X \xrightarrow[f]{2:1} \mathbb{P}^{(1,2,1)}(\mathcal{O}_Y(2h) \oplus \mathcal{O}_Y(f + 2h) \oplus \mathcal{O}_Y(f)) \overset{\varphi_{\widetilde{\kappa}}}{\twoheadrightarrow} Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \twoheadrightarrow \mathbb{P}^1$$

We finally observe that, accordingly with Proposition 2.32, the fan Σ admits only the following primitive collections

$$\mathcal{P} = \{\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}, \quad \mathcal{P}' = \{\mathbf{v}_3, \mathbf{v}_4\}, \quad \mathcal{P}'' = \{\mathbf{v}_1, \mathbf{v}_2\}. \tag{32}$$

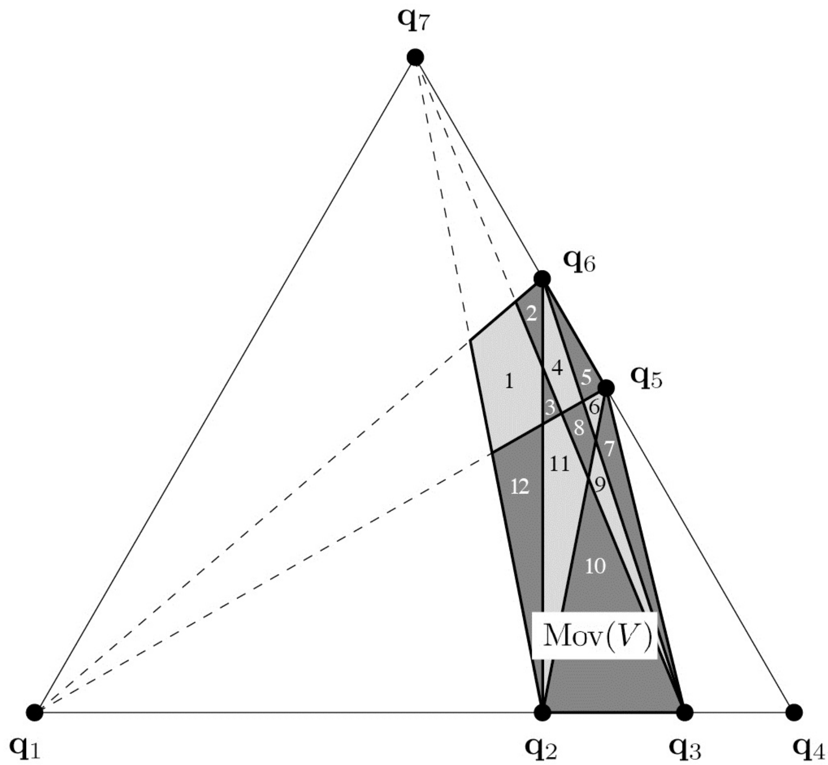

Figure 5: Example 2.44: the section of the cone Mov(V) and its twelve chambers, inside the Gale dual cone $\mathcal{Q} = F_+^3$, as cut out by the plane $\sum_{i=1}^3 x_i^2 = 1$.

Example 2.44. The present example gives an account of Case (c) in the proof of Theorem 2.22. Moreover this example is obtained from Example 2.43 by *breaking the symmetry* around the ray $\left\langle \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \right\rangle \in \Gamma(1)$ of the secondary fan Γ. This is enough to get a simplicial and complete fan $\Sigma \in \mathcal{SF}(V)$ such that $\gamma_\Sigma = \mathrm{Nef}(X(\Sigma)) = 0$.

Consider a 4-dimensional PWS of rank 3 given by the following reduced fan and weight matrices

$$V = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 6 & -12 \\ 0 & 1 & -1 & 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \quad \Longrightarrow \quad Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} = \mathcal{G}(V)$$

Then the following assertions hold:

- $|\mathbb{P}\mathcal{SF}(V)| = 12 < 13 = |\mathcal{SF}(V)|$, meaning that V carries one fan of a $\mathbb{Q}$-factorial complete toric variety of rank 3 which is not projective; explicitly this is generated as the faces of maximal cones in

$$\begin{aligned} \Sigma_{13}(4) = \{&\{3,4,5,7\}, \{3,4,6,7\}, \{2,3,5,7\}, \{2,3,5,6\}, \{2,3,6,7\}, \{2,4,5,7\}, \\ &\{2,4,6,7\}, \{1,3,4,5\}, \{1,3,5,6\}, \{1,3,4,6\}, \{1,2,4,5\}, \{1,2,5,6\}, \{1,2,4,6\}\}. \end{aligned}$$

  The intersection of the cones in the associated bunch of cones inside $\mathcal{Q}$ gives the trivial cone $\langle 0 \rangle$.
- $X(\Sigma)$ is singular for every fan $\Sigma \in \mathcal{SF}(V)$,
- among the 12 distinct chambers of Mov(V) giving the projective fans, there are two maxbord chambers which are both not recursively maxbord chambers: these two are given by the simplicial chambers (see Figure 5)

$$\gamma_5 := \langle \mathbf{q}_5, \mathbf{q}_6, \mathbf{w}_1 \rangle = \left\langle \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 12 \\ 1 & 2 & 12 \end{matrix} \right\rangle, \quad \gamma_{10} := \langle \mathbf{q}_2, \mathbf{q}_3, \mathbf{w}_2 \rangle = \left\langle \begin{matrix} 1 & 1 & 1 \\ 2 & 6 & 6 \\ 0 & 0 & 4 \end{matrix} \right\rangle$$

which give the fans of faces of maximal cones in the following lists

$$\begin{aligned}\Sigma_5(4) = \{&\{2,3,5,7\},\{2,3,4,7\},\{2,3,5,6\},\{2,3,4,6\},\{1,3,5,7\},\{1,3,4,7\},\\ &\{1,3,5,6\},\{1,3,4,6\},\{1,2,5,7\},\{1,2,4,7\},\{1,2,5,6\},\{1,2,4,6\}\}\\ \Sigma_{10}(4) = \{&\{2,3,5,7\},\{2,3,5,6\},\{2,3,6,7\},\{2,4,5,7\},\{2,4,5,6\},\{2,4,6,7\},\\ &\{1,3,5,7\},\{1,3,5,6\},\{1,3,6,7\},\{1,4,5,7\},\{1,4,5,6\},\{1,4,6,7\}\},\end{aligned}$$

- γ_5 is maxbord with respect to the hyperplane H_1 and γ_{10} is maxbord with respect to the hyperplane H_3,
- the maxbord hyperplane H_1 gives the nef primitive collection $\mathcal{P}' = \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ for $\Sigma_5 = \Sigma_{\gamma_5}$ whose numerically effective primitive relation $\kappa' = r_{\mathbb{Z}}(\mathcal{P}')$ gives the first row of Q,
- the maxbord hyperplane H_3 gives the nef primitive collection $\mathcal{P} = \{\mathbf{v}_5,\mathbf{v}_6,\mathbf{v}_7\}$ for $\Sigma_{10} = \Sigma_{\gamma_{10}}$ whose numerically effective primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P})$ gives the last row of Q.

We start by studying $X(\Sigma_{10})$. The left-upper 2×4 submatrix $Q' = \begin{pmatrix}1&1&1&0\\0&2&6&2\end{pmatrix}$ is a positive matrix which does not satisfy Condition (b) in the Definition 1.3: we are then in Case (c) in the proof of Theorem 2.22. Using the same notation therein, we get

$$A = \mathrm{diag}(1,1/2,1/2) \in \mathrm{GL}_3(\mathbb{Q}), \qquad B = \mathrm{diag}(1,1,1,1,2,2,2) \in \mathrm{GL}_7(\mathbb{Q}) \cap \mathbf{M}_7(\mathbb{Z})$$

which give

$$\widetilde{Q} = AQB = \begin{pmatrix}1&11&00&00\\0&13&11&10\\0&00&01&21\end{pmatrix}, \qquad \widetilde{V} = \mathcal{G}(\widetilde{Q}) = \begin{pmatrix}1&0-1&00&3-6\\0&1-1&00&2-4\\0&00&10&-12\\0&00&01&-11\end{pmatrix}.$$

As in the previous Example 2.41, let $U \in \mathrm{GL}_7(\mathbb{Z})$ such that $U\cdot V^T = H = \begin{pmatrix}I_4\\\mathbf{0}_{3,4}\end{pmatrix}$. Then

$$V^T\cdot C = B\cdot\widetilde{V}^T \quad\Longrightarrow\quad C = {}^4U\cdot B\cdot\widetilde{V}^T = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2\end{pmatrix}.$$

Since $\det C = 2$, denoting by $\overline{f}: N \to \widetilde{N}$ the $\mathbb{Z}$-linear map represented by C^T, one gets that $\overline{f}(N)$ is a subgroup of index 2 in $\widetilde{N}$. Consider the fan $\widetilde{\Sigma} = \overline{f}_{\mathbb{R}}(\Sigma_{10})$ and the associated $\mathbb{Q}$-factorial projective toric variety $\widetilde{X}(\widetilde{\Sigma})$. The left-upper 2×4 submatrix $\widetilde{Q}' = \begin{pmatrix}1&1&1&0\\0&1&3&1\end{pmatrix}$ is now a W-matrix, meaning that $\widetilde{Q}$ satisfies Condition (a) in the proof of Theorem 2.22, and $\widetilde{X}$ is either a WPTB or a WPTwB. Subtracting the third row from the second one, we get

$$\widetilde{Q} \sim \begin{pmatrix}1&1&1&0&0&0&0\\0&1&3&1&0&-1&-1\\0&0&0&0&1&2&1\end{pmatrix}$$

and $\widetilde{X}$ is a WPTB if and only the columns of the right upper 2×3 submatrix $\widetilde{Q}'' = \begin{pmatrix}0&0&0\\0&-1&-1\end{pmatrix}$ belongs to $\mathrm{Pic}(Y)$, where Y is the toric surface of rank 2 determined by the fan associated with unique chamber of

$$\mathrm{Mov}(\mathcal{G}(\widetilde{Q}')) = \left\langle\begin{matrix}1&1\\1&3\end{matrix}\right\rangle.$$

Recalling [22, Theorem 2.9(2)] a basis of $\mathrm{Pic}(Y)$ can be determined by following the procedure described in [22, § 1.2.3]; we obtain:

$$\mathrm{Pic}(Y) = \mathcal{L}(L_1,L_2) \cong \mathbb{Z}^2 \quad \text{where } L_1 = \begin{pmatrix}2\\0\end{pmatrix}, L_2 = \begin{pmatrix}-1\\3\end{pmatrix}.$$

Hence $-(L_1+2L_2) = \left(\begin{smallmatrix}0\\-6\end{smallmatrix}\right)$, meaning that the Cartier indices of Weil divisors whose classes are the columns of $\widetilde{Q}''$ are $l_0 = 1, l_1 = l_2 = 6$, respectively. Recalling Proposition 2.19, $\widetilde{X}$ turns out to be a WPTwB and in particular a toric cover of the WPTB $\mathbb{P}^{W'}(\mathcal{E})$ where $W' = W = (1,2,1)$ is the reduced weight vector of $(l_0w_0, l_1w_1, l_2w_2) = (1,12,6)$ and $\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(6D_4')^{\oplus 2}$.

To explicitly determine the toric cover $g : \widetilde{X} \to \mathbb{P}^W(\mathcal{E})$ one has to determine matrices Δ, Λ, Φ as in the proof of Proposition 2.19. Namely

$$\Delta = \operatorname{diag}(1,1,1/6) \in \mathrm{GL}_3(\mathbb{Q}), \qquad \Lambda = \operatorname{diag}(1,1,1,1,6,6,6) \in \mathrm{GL}_7(\mathbb{Q}) \cap \mathbf{M}_7(\mathbb{Z})$$

which give

$$\widetilde{\widetilde{Q}} = \Delta\widetilde{Q}\Lambda = \begin{pmatrix}1&1&1&0&0&0&0\\0&1&3&1&6&6&0\\0&0&0&0&1&2&1\end{pmatrix}, \qquad \widetilde{\widetilde{V}} = \mathcal{G}\left(\widetilde{\widetilde{Q}}\right) = \begin{pmatrix}1&0&-1&3&0&0&0\\0&1&-1&2&0&0&0\\0&0&0&6&0&-1&2\\0&0&0&0&1&-1&1\end{pmatrix}.$$

Hence

$$\mathbb{P}^W(\mathcal{E}) = \mathbb{P}^{(1,2,1)}\left(\mathcal{O}_Y \oplus \mathcal{O}_Y(6D_4')^{\oplus 2}\right) = \widetilde{\widetilde{X}}\left(\widetilde{\widetilde{\Sigma}}\right), \text{ with } \widetilde{\widetilde{\Sigma}} = \overline{g}_{\mathbb{R}}\left(\widetilde{\Sigma}\right) = \overline{(g\circ f)}_{\mathbb{R}}(\Sigma_{10}).$$

Moreover, choosing $\widetilde{U} \in \mathrm{GL}_7(\mathbb{Z})$ such that $\widetilde{U}\cdot\widetilde{V}^T = \widetilde{H} = \begin{pmatrix}I_4\\ \mathbf{0}_{3,4}\end{pmatrix}$ one obtains

$$\widetilde{V}^T\cdot\Phi = \Lambda\cdot\widetilde{\widetilde{V}}^T \quad\Longrightarrow\quad \Phi = {}^4\widetilde{U}\cdot\Lambda\cdot\widetilde{\widetilde{V}}^T = \begin{pmatrix}1&0&0&0\\0&1&0&0\\3&2&6&0\\0&0&0&6\end{pmatrix}.$$

Then $\overline{g}\left(\widetilde{N}\right)$ is a subgroup of index $\det(\Phi) = 36$ of $\widetilde{\widetilde{N}}$. Therefore $X(\Sigma_{10})$ turns out to admit the following geometric structure:

$$X(\Sigma_{10}) \xrightarrow[f]{2:1} \widetilde{X} \xrightarrow[g]{36:1} \mathbb{P}^{(1,2,1)}\left(\mathcal{O}_Y \oplus \mathcal{O}_Y(6D_4')^{\oplus 2}\right) \xrightarrow[\varphi_{\widetilde{\widetilde{\kappa}}}]{} Y$$

where

- f is a double toric cover ramified along the torus invariant Weil divisors D_5, D_6, D_7 of X, as one can immediately deduce from the diagonal matrix B,
- g is a $36:1$ toric cover ramified along the torus invariant Weil divisors $\widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7$ of $\widetilde{X}$, as one can immediately deduce from the diagonal matrix Λ,
- $\varphi_{\widetilde{\widetilde{\kappa}}}$ is the contraction morphism of the contractible class $\widetilde{\widetilde{\kappa}} = g\circ f(\kappa)$, under the notation introduced in Definition 2.37, meaning that κ is a pseudo-contractible class.

For what concerns the further maxbord chamber γ_5, the study of $X(\Sigma_5)$ proceeds in the same way as for $X(\Sigma_{10})$, after exchanging with each other the first and third rows of Q and reordering the columns to still get a REF matrix, hence obtaining

$$Q \sim \begin{pmatrix}1&2&1&0&0&0&0\\0&1&1&2&6&2&0\\0&0&0&0&1&1&1\end{pmatrix}. \tag{33}$$

Let us reassign Q as the right matrix in (33). Then we get an analogous reassignment for

$$V = \mathcal{G}(Q) = \begin{pmatrix}1&1&-3&0&0&1&-1\\0&2&-4&0&0&1&-1\\0&0&0&1&0&-1&1\\0&0&0&0&1&-3&2\end{pmatrix}.$$

Now the left upper 2×4 submatrix $Q' = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ is a W-matrix, implying that $X(\Sigma_5)$ is already either a WPTB or a WPTwB over the toric surface Y' of rank 2 and determined by the fan associated with unique chamber of

$$\mathrm{Mov}(\mathcal{G}(\widetilde{Q}')) = \left\langle \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix} \right\rangle .$$

Still applying [22, Theorem 2.9(2)] we get

$$\mathrm{Pic}(Y') = \mathcal{L}(L_1, L_2) \cong \mathbb{Z}^2 \quad \text{where } L_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

By subtracting 6 times the third row from the second one in Q we get

$$Q \sim \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and we see that the column of the right upper 2×3 submatrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -6 \end{pmatrix}$ belongs to $\mathrm{Pic}(Y')$. Then $X(\Sigma_5)$ is a WPTB over Y' and better

$$X(\Sigma_5) = \mathbb{P}\left(\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(2D'_4) \oplus \mathcal{O}_{Y'}(3D'_4)\right) \xrightarrow{\ \varphi_{\kappa'}\ } Y'$$

is actually a PTB over Y', whose fibers are isomorphic to $\mathbb{P}^2$ since $W = (1, 1, 1)$: the fibration morphism $\varphi_{\kappa'}$ is given by the contraction of the contractible class $\kappa' = r_{\mathbb{Z}}(\mathcal{P}')$.

For the remaining ten fans Σ_i with $i = 1, \dots, 4, 6, \dots, 9, 11, 12$, by Theorem 2.24 we can only say that $X(\Sigma_i)$ is a toric flip either of $X(\Sigma_{10})$, hence of a toric cover of a WPTB, or of $X(\Sigma_5)$, hence of a PTB.

Remark 2.45. For smooth threefolds Fujino and Payne [11] proved that $\mathrm{Nef}(X) \neq 0$ for $r \leq 4$. In Example 2.44 the fan Σ_{13} is associated with the trivial cone $\langle 0 \rangle \subset \mathcal{Q}$, giving a 4-dimensional $\mathbb{Q}$-factorial complete toric variety X with Picard number $r = 3$ such that $\mathrm{Nef}(X) = 0$, hence showing that the Fujino-Payne inequality does no more hold when dropping the smoothness hypothesis. One might object that the given example has dimension 4, while the result of Fujino and Payne applies in dimension 3, but in the forthcoming paper [24] we provide an example of a 3-dimensional $\mathbb{Q}$-factorial complete toric variety X with Picard number $r = 3$ such that $\mathrm{Nef}(X) = 0$. Anyway we claim that example to be sharp, both for the dimension n and the rank r, since, on the one hand, it is well known that every complete toric variety of dimension at most 2 is projective (see e.g. [18, § 8, Proposition 8.1]) and, on the other hand, in the same paper we prove that *a $\mathbb{Q}$-factorial, complete, toric variety of rank $r = 2$ is projective.*

3 Classifying $\mathbb{Q}$-factorial projective toric varieties

In this section we apply the results obtained in § 2 for a PWS to the case of a singular $\mathbb{Q}$-factorial projective variety, recalling that the latter is always a finite quotient of a PWS, after [23, Theorem 2.2].

toric

3.1 1-coverings of $\mathbb{Q}$-factorial complete tric varieties. For the reader's convenience we present definitions and results from [23] that are needed in § 3.2. For ease, let us here assume that X and Y are normal and complete algebraic varieties, which is enough for our purpose.

Definition 3.1 (see Definition 1.9 in [23]). A finite surjective morphism $\varphi : Y \to X$ is called a *covering in codimension* 1 (or simply a 1-*covering*) if it is unramified in codimension 1, that is, there exists a subvariety $V \subseteq X$ such that $\mathrm{codim}_X V \geq 2$ and $\varphi|_{Y_V}$ is a topological covering, where $Y_V := \varphi^{-1}(X \setminus V)$. Moreover a *universal covering in codimension* 1 is a 1-covering $\varphi : Y \to X$ such that for any 1-covering $\phi : X' \to X$ of X there exists a 1-covering $f : Y \to X'$ with $\varphi = \phi \circ f$.

Recall notation introduced in Definitions 1.2, 1.3 and 1.4. A $\mathbb{Q}$-factorial complete toric variety X is given by a reduced F-matrix V and a fan $\Sigma \in \mathcal{SF}(V)$ such that $X = X(\Sigma)$. Let $Q = \mathcal{G}(V)$ be a positive REF W-matrix. Then, by [22, Proposition 3.12(1)], $\widehat{V} = \mathcal{G}(Q)$ is a CF-matrix and the choice of $\Sigma \in \mathcal{SF}(V)$ uniquely determines a fan $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$ such that $Y = Y(\widehat{\Sigma})$ is a PWS, so giving a canonical universal 1-covering of X. This is, in short, one of the main results of [23], namely:

Theorem 3.2 (Theorem 2.2 in [23]). *Every $\mathbb{Q}$-factorial, complete toric variety X admits a canonical universal 1-covering Y which is a PWS such that the 1-covering morphism $\varphi : Y \to X$ is equivariant with respect to the torus actions. In particular, every $\mathbb{Q}$-factorial, complete toric variety X can be canonically described as a finite geometric quotient $X \cong Y/\pi_1(X_{reg})$ of a PWS Y by the torus-equivariant action of $\pi_1(X_{reg}) \cong \mathrm{Tors}(\mathrm{Cl}(X))$.*

In particular, if X is projective then, by construction, the fans Σ and $\widehat{\Sigma}$ are associated with the same chamber, i.e.

$$\gamma_\Sigma = \gamma_{\widehat{\Sigma}} \subseteq \mathrm{Mov}(V) \subseteq \mathcal{Q} = \langle Q \rangle .$$

Remark 3.3. The action of $\pi_1(X_{\mathrm{reg}})$ on Y can be (non-canonically) described by means of a *torsion matrix Γ* representing the *torsion part* of the class morphism

$$d_X = f_X \oplus \tau_X : \mathcal{W}_T(X) \xrightarrow{Q \oplus \Gamma} \mathrm{Cl}(X) \cong F \oplus \mathrm{Tors}(\mathrm{Cl}(X))$$

where F is a free part of the class group $\mathrm{Cl}(X)$. Namely:

(1) The torsion matrix Γ is constructed as follows:
- choose fan matrices V and $\widehat{V} = \mathcal{G}(\mathcal{G}(V))$ of X and Y, respectively, such that there exists a diagonal matrix $\Delta = \mathrm{diag}(c_1, \dots, c_n) \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ with
  - $1 = c_1 \mid \dots \mid c_n$,
  - $V = \Delta \cdot \widehat{V}$,
  - $\mathrm{Tors}(\mathrm{Cl}(X)) \cong \bigoplus_{i=1}^n \mathbb{Z}/c_i\mathbb{Z} = \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$, according to the decomposition of $\mathrm{Cl}(X)$ given by the fundamental theorem of finitely generated abelian groups,

  $$\mathrm{Cl}(X) = F \oplus \mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$$

  where $s < n$, $\tau_k = c_{n-s+k} > 1$, $c_1 = \dots = c_{n-s} = 1$

  (This is possible by [23, Theorem 3.2(4)]);
- recalling notation on submatrices given in list 1.1, consider

$$\begin{aligned} U_Q \in \mathrm{GL}_{n+r}(\mathbb{Z}) : &\quad U_Q \cdot Q^T = \mathrm{HNF}(Q^T) \\ U := \begin{pmatrix} {}^r U_Q \\ \widehat{V} \end{pmatrix} &\in \mathrm{GL}_{n+r}(\mathbb{Z}) \\ W \in \mathrm{GL}_{n+r}(\mathbb{Z}) : &\quad W \cdot ({}^{n+r-s}U)^T = \mathrm{HNF}\left(({}^{n+r-s}U)^T\right) \\ G := {}_s\widehat{V} \cdot ({}_sW)^T &\in \mathbf{M}_s(\mathbb{Z}) \\ U_G \in \mathrm{GL}_s(\mathbb{Z}) : &\quad U_G \cdot G^T = \mathrm{HNF}(G^T) \end{aligned}$$

  and then define

$$\Gamma = U_G \cdot {}_sW \mod \boldsymbol{\tau} \tag{34}$$

  where this notation means that the (k, j)-entry of Γ is given by the class in $\mathbb{Z}/\tau_k\mathbb{Z}$ represented by the corresponding (k, j)-entry of ${}^sU_G \cdot {}_sW$, for $1 \leq k \leq s$ and $1 \leq j \leq n + r$; see [26, Theorem 3.2(6)].

(2) Consider the action of $\pi_1(X_{\mathrm{reg}})$ defined by means of its dual group $\mu(X) := \mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*)$ and induced by the natural complex multiplication of $\mathrm{Hom}(\mathcal{W}_T(Y), \mathbb{C}^*)$ on Y and the injection $\mu(X) \hookrightarrow \mathrm{Hom}(\mathcal{W}_T(Y), \mathbb{C}^*)$ dually determined by Γ; see [23, § 4].

Such an action gives rise to a good geometric quotient $Y \twoheadrightarrow X = Y/\mu$, due to the famous result of Cox [8].

3.2 A Batyrev type classification. Theorem 3.2, jointly with Theorem 2.22, Theorem 2.24 and Theorem 2.28, allows us to prove the following statements.

Theorem 3.4. *Given a reduced $n \times (n+r)$ F-matrix V with $r \geq 2$, a chamber $\gamma \in \mathcal{A}_\Gamma(V)$ is maximally bordering if and only if the associated $\mathbb{Q}$-factorial projective toric variety $X(\Sigma_\gamma)$ is a finite abelian quotient of a toric cover $Y(\widehat{\Sigma})$ of a weighted projective toric bundle $\mathbb{P}^W(\mathcal{E})$. In particular, the quotient map $Y \twoheadrightarrow X$ gives a Galois covering ramified in codimension ≥ 2, whose Galois group is $\mu(X)$ and described as above by a torsion matrix Γ determined as in* (34).

Remark 3.5. Recalling § 2.4 and Remark 2.39, given a $\mathbb{Q}$-factorial projective toric variety X whose fan is associated with a maxbord chamber we find the following situation:

$$\begin{tikzcd}
X = Y/\mu & Y \arrow[l, "\text{universal}\atop\text{1-covering}"'] \arrow[r, "\text{toric}\atop\text{cover}", "f"'] \arrow[d, "\phi", "\text{fake WPS}\atop\text{fibering}"'] & \mathbb{P}^W(\mathcal{E}) \arrow[d, "\varphi"', "\text{WPTB}"] \\
& X_0 \arrow[r, "f_0", "\text{finite}"'] & X'
\end{tikzcd}$$

In particular, starting from a fan matrix V of X, both the universal 1-covering and the right hand side composition of toric morphisms $\varphi \circ f$ are explicitly described.

Remark 3.6. Note that Theorem 3.4 provides a definitive answer to the question left open in Remark 2.20, about the the geometric interpretation of the toric variety $X(\Sigma)$ constructed from a fan Σ generated by fibred cones as in Proposition 2.16 and admitting a fan matrix as in (15) which is an F non-CF matrix.

Theorem 3.7. *Let V be a reduced $n \times (n+r)$ F-matrix and let $X(\Sigma)$ be a $\mathbb{Q}$-factorial projective toric variety, with $\Sigma \in \mathbb{P}\mathcal{SF}(V)$. Then X is a toric flip of a finite abelian quotient X' of a toric cover $Y' \twoheadrightarrow \mathbb{P}^W(\mathcal{E})$ of a WPTB if and only if* $\mathrm{Mov}(V)$ *is maxbord with respect to a hyperplane $H \subseteq F^r_{\mathbb{R}}$. In particular:*

(1) *calling Y the PWS giving the universal* 1*-covering of X, as in Theorem* 3.2*, the toric flip $X \dashrightarrow X'$ uniquely lifts to giving a toric flip $Y \dashrightarrow Y'$ between* 1*-coverings and giving rise to the following commutative diagram*

$$\begin{tikzcd}
Y \arrow[r, dashed] \arrow[d, twoheadrightarrow] & Y' \arrow[d, twoheadrightarrow] \\
X \arrow[r, dashed] & X'
\end{tikzcd}$$

in which vertical maps represent Galois coverings ramified in codimension ≥ 2, both with Galois group $\mu(X) = \mu(X')$ and both described by the same torsion matrix Γ determined as in (34)*;*

(2) *X' has associated chamber $\gamma' \subseteq$* $\mathrm{Mov}(V) \subseteq \mathcal{Q}$ *which is maxbord with respect to the hyperplane H;*

(3) *the finite abelian quotient $Y' \twoheadrightarrow X'$ is described by Theorem* 3.4.

Theorem 3.8. *A $\mathbb{Q}$-factorial projective variety $X(\Sigma)$ is a finite abelian quotient of a PWS, say Y, which is produced from a toric cover of a WPS by a sequence of toric covers of WPTB's if and only if the corresponding chamber γ_Σ is recursively maxbord. In particular, the quotient map $Y \twoheadrightarrow X$ gives a Galois covering ramified in codimension ≥ 2, whose Galois group is $\mu(X)$ and described by a torsion matrix Γ determined as in* (34).

As above, let V be a reduced F-matrix, let $Q = \mathcal{G}(V)$ be a positive, REF, W-matrix and let $\widehat{V} = \mathcal{G}(Q)$ be a CF-matrix. Let $X(\Sigma)$ be the $\mathbb{Q}$-factorial projective toric variety given either by the choice of a fan $\Sigma \in \mathbb{P}\mathcal{SF}(V)$ or by the choice of a chamber $\gamma = \gamma_\Sigma \in \mathcal{A}_\Gamma(V)$. Let $Y(\widehat{\Sigma})$ be the PWS giving the universal 1-covering of X, which is $\widehat{\Sigma} = \widehat{\Sigma}_\gamma \in \mathbb{P}\mathcal{SF}(\widehat{V})$. Let $\mathcal{P} = \{V_P\}$, for some $P \subseteq \{1, \dots, n+r\}$, be a nef primitive collection for Σ. Then $\widehat{\mathcal{P}} := \{\widehat{V}_P\}$ is such that $\mathcal{P}^* = \widehat{\mathcal{P}}^* \in \mathcal{Q}(1)$ meaning that $\widehat{\mathcal{P}}$ is a nef primitive collection for $\widehat{\Sigma}$ if and only if $\mathcal{P}$ is a nef primitive collection for Σ. Then

$\kappa := r_{\mathbb{Z}}(\mathcal{P}) \in A_1(X) \cap \overline{\mathrm{NE}}(X)$ is a numerically effective primitive relation for Σ if and only if $\widehat{\kappa} := r_{\mathbb{Z}}(\widehat{\mathcal{P}}) \in A_1(Y) \cap \overline{\mathrm{NE}}(Y)$ is a numerically effective primitive relation for $\widehat{\Sigma}$.

In the following, the class $\widehat{\kappa}$ is called *the universal lifting of κ*.

Theorem 3.9. *Let V be a reduced $n \times (n+r)$ F-matrix and $\Sigma \in \mathbb{P}\mathcal{SF}(V)$. Assume that there exists a primitive collection $\mathcal{P}$ for Σ whose primitive relation $\kappa = r_{\mathbb{Z}}(\mathcal{P})$ is numerically effective. Then the universal lifting $\widehat{\kappa}$ of κ is either contractible or pseudo-contractible if and only if one of the following equivalent conditions occurs:*

(I) *γ_Σ is a maxbord chamber,*
(II) *for every primitive collection $\mathcal{P}' \neq \mathcal{P}$, for Σ, then $\mathcal{P}' \cap \mathcal{P} = \emptyset$.*

In particular, $\widehat{\kappa}$ is either contractible or pseudo-contractible depending on which condition in Theorem 2.38 is satisfied.

Example 3.10. We consider the following 4×7 matrix

$$V = \begin{pmatrix} 9 & 11 & 13 & -33 & 9 & 44 & -97 \\ 10 & 12 & 14 & -36 & 10 & 48 & -106 \\ 54 & 63 & 75 & -192 & 51 & 258 & -567 \\ 310 & 365 & 430 & -1105 & 295 & 1485 & -3265 \end{pmatrix}.$$

First we need to understand if V is a F-matrix: if this is the case, then V is a reduced F-matrix since the gcd of entries in every column is always 1.

A matrix $U_V \in \mathrm{GL}_7(\mathbb{Z})$ such that $\mathrm{HNF}(V^T) = U_V \cdot V^T$ is given by

$$U_V = \begin{pmatrix} -4 & 0 & 1 & -1 & -1 & 0 & 0 \\ 9 & 2 & 5 & 7 & 7 & 0 & 0 \\ 3 & -4 & 5 & 2 & 2 & 0 & 0 \\ -1 & 1 & -2 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 1 & 0 \\ 3 & 3 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}$$

whose bottom three rows give the matrix

$${}_3U_V = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 1 & 0 \\ 3 & 3 & 1 & -1 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} =: Q.$$

Note that the equivalence ${}_3U_V \sim Q$ is realized by means of the matrix

$$M = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}) : \quad Q = M \cdot {}_3U_V.$$

Since $Q = \mathcal{G}(V)$ is a reduced W-matrix, V is a reduced F-matrix by [22, Proposition 3.12(2)]. In particular Q is the same positive REF W-matrix as in Example 2.43, meaning that $\widehat{V} = \mathcal{G}(Q)$ is given by the matrix V in Example 2.43. Therefore we have a unique maxbord chamber given by

$$\gamma_8 = \langle \mathbf{q}_1 = \mathbf{q}_2, \mathbf{q}_3, \mathbf{w} \rangle = \left\langle \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\rangle$$

which is also a recursively maxbord chamber: we are then in the situation described by Theorem 3.8. Calling $\Sigma \in \mathbb{P}\mathcal{SF}(V)$ and $\widehat{\Sigma} \in \mathbb{P}\mathcal{SF}(\widehat{V})$ the corresponding fans, the covering $Y(\widehat{\Sigma})$ is given by the PWS $X(\Sigma_8)$ described in Example 2.43, i.e.

$$Y \xrightarrow[f]{2:1} \mathbb{P}^{(1,2,1)}\left(\mathcal{O}_S(2h) \oplus \mathcal{O}_S(f+2h) \oplus \mathcal{O}_S(f)\right) \overset{\varphi_{\widehat{\kappa}}}{\twoheadrightarrow} S \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) \twoheadrightarrow \mathbb{P}^1$$

where f, h are the generators of $\mathrm{Pic}(S)$ given by the fibre and the pull-back of the Picard generator $\mathcal{O}_{\mathbb{P}^1}(1)$ of the base $\mathbb{P}^1$, respectively, and $\widetilde{\kappa} = f(\kappa)$ is the contractible class image of the pseudo-contractible class κ which is the numerically effective primitive relation given by the bottom row of Q: recalling Theorem 3.9, κ is the universal lifting of the primitive relation $r_{\mathbb{Z}}(\mathcal{P})$, associated with the nef primitive collection $\mathcal{P} = \{\mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$ for Σ.

We give here a better description of the toric cover $f : Y \overset{2:1}{\twoheadrightarrow} \mathbb{P}^W(\mathcal{E})$, where $W = (1, 2, 1)$ and $\mathcal{E} = \mathcal{O}_S(2h) \oplus \mathcal{O}_S(f+2h) \oplus \mathcal{O}_S(f)$. Both Y and $\mathbb{P}^W(\mathcal{E})$ are PWS, meaning that they can be completely described as Cox geometric quotients by means of the W-matrices Q and $\widetilde{Q}$ given in Example 2.43. We denote by $Z \subseteq \mathbb{C}^7$ the zero-locus of the irrelevant ideal associated with the fan Σ (see [8] for further details). Then

- Y is the geometric quotient obtained by the following action of $(\mathbb{C}^*)^3$
$$g : (\mathbb{C}^*)^3 \times (\mathbb{C}^7 \setminus Z) \longrightarrow (\mathbb{C}^7 \setminus Z)$$
defined by setting
$$g((t_1, t_2, t_3), (x_1, \dots, x_7)) := (t_1x_1, t_1x_2, t_1t_2x_3, t_1t_2^2x_4, t_2t_3x_5, t_2t_3^2x_6, t_3x_7);$$
- $\mathbb{P}^W(\mathcal{E})$ is the geometric quotient obtained by the following action of $(\mathbb{C}^*)^3$
$$l : (\mathbb{C}^*)^3 \times (\mathbb{C}^7 \setminus Z) \longrightarrow (\mathbb{C}^7 \setminus Z)$$
defined by setting
$$l((s_1, s_2, s_3), (y_1, \dots, y_7)) := (s_1y_1, s_1y_2, s_1^2s_2y_3, s_1s_2^2y_4, s_2s_3y_5, s_2s_3^2y_6, s_3y_7);$$
- calling $[X_1 : \dots : X_7]$ and $[Y_1 : \dots : Y_7]$ the associated homogeneous coordinates on Y and $\mathbb{P}^W(\mathcal{E})$, respectively, and recalling the matrices A^{-1} and B in (30), the toric cover f is given by setting $Y_i = X_i$ for $i = 1, 2, 4$ and $Y_j = X_j^2$ for $j = 3, 5, 6, 7$. One can easily check that the latter is consistent with the given actions g and l.

Finally we need to describe the finite quotient $Y \twoheadrightarrow X$, to complete the geometric description of the $\mathbb{Q}$-factorial projective variety $X(\Sigma)$ as given in Theorem 3.4. For this purpose we have to determine the torsion matrix Γ as in (34). Then

$$H = \mathrm{HNF}(V) = \begin{pmatrix} 1 & 0 & 0 & -1 & 10 & -8 & 6 \\ 0 & 1 & 0 & -1 & 27 & -25 & 23 \\ 0 & 0 & 1 & -1 & 24 & -23 & 22 \\ 0 & 0 & 0 & 0 & 30 & -30 & 30 \end{pmatrix}$$

$$U = \begin{pmatrix} -13 & 36 & 7 & -2 \\ -26 & 92 & 16 & -5 \\ -22 & 83 & 17 & -5 \\ -30 & 105 & 20 & -6 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}) : \quad U \cdot V = H$$

$$\widehat{H} = \mathrm{HNF}(\widehat{V}) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} = \widehat{V} \quad \Longrightarrow \quad \widehat{U} = I_4.$$

By [22, Proposition 3.1(3)] there exist $\beta, \beta_H \in \mathbf{M}(4, 4; \mathbb{Z}) \cap \mathrm{GL}(4, \mathbb{Q})$ such that $V = \beta\widehat{V}$ and $H = \beta_H\widehat{H}$, namely

$$\beta_H = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 27 \\ 0 & 0 & 1 & 24 \\ 0 & 0 & 0 & 30 \end{pmatrix} \quad \Longrightarrow \quad \beta = U^{-1} \cdot \beta_H \cdot \widehat{U} = \begin{pmatrix} 9 & 11 & 13 & 9 \\ 10 & 12 & 14 & 10 \\ 54 & 63 & 75 & 51 \\ 310 & 365 & 430 & 295 \end{pmatrix}.$$

Therefore $\Delta = \mathrm{SNF}(\beta)$ and $\mu, \nu \in \mathrm{GL}_4(\mathbb{Z})$ with $\Delta = \mu \cdot \beta \cdot \nu$ are given by

$$\Delta = \mathrm{diag}(1,1,1,30) \quad \Longrightarrow \quad \mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathbb{Z}/30\mathbb{Z} \quad \text{and} \quad s = 1$$

$$\mu = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 14 & -18 & 1 & 0 \\ 8 & -22 & -3 & 1 \\ -30 & 105 & 20 & -6 \end{pmatrix} \qquad \nu = \begin{pmatrix} 1 & -1 & 4 & 20 \\ 0 & 1 & -5 & -27 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Define

$$\widehat{V}' = \nu^{-1} \cdot \widehat{V} = \begin{pmatrix} 1 & 1 & 1 & -3 & 1 & 4 & -9 \\ 0 & 1 & 5 & -6 & -3 & 10 & -17 \\ 0 & 0 & 1 & -1 & -6 & 7 & -8 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$V' = \mu \cdot V = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 14 & -18 & 1 & 3 & 0 & -7 & 14 \\ 8 & -22 & -3 & 17 & 1 & -32 & 63 \\ -30 & 105 & 20 & -95 & -6 & 176 & -346 \end{pmatrix}.$$

Then $V' = \Delta \widehat{V}'$, as in item (1) of Remark 3.3. A matrix $U_Q \in \mathrm{GL}_7(\mathbb{Z})$ such that $U_Q \cdot Q^T = \mathrm{HNF}(Q^T)$ is given by

$$U_Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -2 & -1 & 0 & 1 & 0 & 0 \\ 3 & -2 & -2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & -2 & 1 & 0 \\ -3 & 2 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

so giving

$$U = \begin{pmatrix} {}^{3}U_Q \\ \widehat{V}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -3 & 1 & 4 & -9 \\ 0 & 1 & 5 & -6 & -3 & 10 & -17 \\ 0 & 0 & 1 & -1 & -6 & 7 & -8 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

The next step is finding $W \in \mathrm{GL}_7(\mathbb{Z})$ such that $W \cdot ({}^{6}U)^T = \mathrm{HNF}(({}^{6}U)^T)$, which is given by

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 46 & -46 & -99 & 46 & 153 & 106 \\ 0 & 2 & -2 & -5 & 2 & 7 & 5 \\ 0 & 38 & -38 & -82 & 38 & 127 & 88 \\ 0 & 47 & -47 & -102 & 47 & 157 & 109 \end{pmatrix}.$$

Since $s = 1$ the matrix G turns out to be the product of the last rows of $\widehat{V}'$ and W, respectively, thus giving $G = U_G = (-1) \in \mathrm{GL}_1(\mathbb{Z})$. Therefore the torsion matrix Γ is obtained by taking the reduction mod 30 of the opposite of the last row of W, that is

$$\Gamma = \begin{pmatrix} [0]_{30} & [13]_{30} & [17]_{30} & [12]_{30} & [13]_{30} & [23]_{30} & [11]_{30} \end{pmatrix}.$$

Thus the finite quotient giving X is obtained by the following action of $\mu_{30} = \mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*)$ on Y:

$$\begin{aligned} k : \mu_{30} \times Y &\longrightarrow Y \\ (\varepsilon, [X_1 : \ldots : X_7]) &\longmapsto [X_1 : \varepsilon^{13} X_2 : \varepsilon^{17} X_3 : \varepsilon^{12} X_4 : \varepsilon^{13} X_5 : \varepsilon^{23} X_6 : \varepsilon^{11}\, X_7]. \end{aligned}$$

Equivalently X can be obtained as a Cox geometric quotient by the following action

$$k \circ g : ((\mathbb{C}^*)^3 \oplus \mu_{30}) \times (\mathbb{C}^7 \setminus Z) \longrightarrow (\mathbb{C}^7 \setminus Z)$$

defined by setting

$$k \circ g((t_1, t_2, t_3), (x_1, \ldots, x_7)) := (t_1 x_1, \varepsilon^{13} t_1 x_2, \varepsilon^{17} t_1 t_2 x_3, \varepsilon^{12} t_1 t_2^2 x_4, \varepsilon^{13} t_2 t_3 x_5, \varepsilon^{23} t_2 t_3^2 x_6, \varepsilon^{11} t_3 x_7)$$

and giving the following geometric picture

$$\begin{tikzcd}
 & \mathbb{C}^7\backslash Z \arrow[r, "\phi"] \arrow[dl, "\pi_k\circ\pi_g"'] \arrow[d, "\pi_g"] & \mathbb{C}^7\backslash Z \arrow[d, "\pi_l"] & & \\
X & Y \arrow[l, two heads, "30:1"', "\pi_k"] \arrow[r, two heads, "2:1", "f"'] & \mathbb{P}^W(\mathcal{E}) \arrow[r, two heads, "\varphi_{\overline{k}}"] & \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) \arrow[r, two heads] & \mathbb{P}^1
\end{tikzcd}$$

where π_g, π_k, π_l are the quotient maps associated with the actions g, k, l, respectively, and the map ϕ is given, recalling (30), by the exponential action of matrices B^T and $(A^{-1})^T$ on the coordinates of $\mathbb{C}^7$ and of the acting $(\mathbb{C}^*)^3$, respectively.

As already observed for the toric cover Y in Example 2.43, also the F-matrix V admits seven further projective and simplicial fans different from Σ, i.e. $|\mathbb{P}\mathcal{SF}(V)| = 8$. By Theorem 3.7, for every $\Sigma' \in \mathbb{P}\mathcal{SF}(V)$ if $\Sigma' \neq \Sigma$ then $X' = X'(\Sigma')$ is a toric flip of $X(\Sigma)$. Moreover, since $\gamma' = \gamma'_{\Sigma'}$ is not a maxbord chamber, Theorem 3.4 guarantees that X' cannot admit a universal 1-covering PWS which is either a WPTB or a toric cover of a WPTB.

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